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Oxidation of Lactic Acid by Hexavalent Chromium

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Abstract

The use of chromic acid as an oxidant of organic-compounds have been largely investigated. This paper describes the kinetics of lactic acid catalysed by hydrogen ion of different concentrations.

The products of the oxidation have been identified and equivalent number for the oxidation of lactic acid have been determined by the oxidant at both the places depending upon the acid range of the medium and temperature. The product of the oxidation may be aldehyde, acetic acid or their mixtures according to the conditions of the experiments.

The oxidation of lactic acid by chromic acid was effected by Chapmann and Smith as early as 1867[1]. Dhar and coworkers[2], reported that the order of reaction is approximately three (two with respect to the acid and one with respect to chromic acid). They have further reported that the reaction can be accelerated by light and catalysed by manganous sulphate[3].

In recent years considerable amount of work has been done on the oxidation of alcohols by Westheimer and coworkers[4] and also by Chatterji and coworkers[5]. The mechanism of this oxidation has been explained on the formation of an estar as an intermediate with alcohol and chromic acid. It has been also generally observed that this oxidation induces the oxidation of manganous salt and therefore bivalent manganese depresses the rate of oxidation of alcohols by chromic acid[6].

Lactic acid has an alcoholic group and also a carboxylic group and it was thought necessary to investigate its kinetics of oxidation by chromic acid in order to clarify as to where the oxidant first attacks this acid. The oxidised products have been identified and the equivalents of hexavalent chromium consumed has also been determined under the same experimental conditions of temperature and concentration of sulphuric acid.

Experimental

Standard solutions of lactic acid, potassium dichromate, sulphuric acid and caustic soda were freshly prepared from A. R. quality reagents. Reaction was carried out in a precision thermostat as described in the earlier paper. The kinetics of the oxidation was followed by estimating hexavalent chromium iodometrically from the aliquot parts of the reaction mixture at different intervals of time.

I shall now record the result of some of the experiments. Over all concentrations of the reactants are noted in all cases.

TABLE 1

Effect of sulphuric acid :

Concentration of Lactic acid = 0·188 M

Concentration of potassium dichromate = 0·0025 N

Temp. 30°C

Time in mts.	Concentration of sulphuric acid in reaction mixture			
	Nil $k_1 = K_1 \times 10^{-3}$	·02N	·05N	0·1N
0	–	–	–	–
5	5·38	11·00	16·82	18·74
15	8·18	12·65	17·65	20·77
30	8·39	12·32	16·59	20·19
45	8·46	12·56	16·37	18·70
60	8·53	12·57	15·97	17·42
75	8·41	12·28	–	–
90	8·49	–	–	–
Average value =	8·41	12·48	16·68	19·16

The first order constants for the oxidant are recorded in the above table. It should be mentioned here that where no sulphuric acid were added the constants in the initial stage were small showing that the reaction tends to show an induction period. It is further seen that the rate increases with the increasing concentration of sulphuric acid and is approximately proportional to the square-root of the concentration of sulphuric acid.

The effect of lower concentration of Hydrogen ion has been also investigated be nutralizing lactic acid with known amount of caustic soda solution. The pH of lactic acid and lactate solutions were measured by Beckman pH meter. The reaction mixtures of lactic acid and lactate solution used were of pH 3·38, 3·82, 4·30, 5·00. The results show that induction period increases with increasing pH of reacting lactate-lactic acid solution and the reaction becomes exceeding slow above pH 5.

Total order of the reaction

The order of the reaction with respect to lactic acid was determined by isolation method for the reaction in the presence of different amount of sulphuric acid. The first order constants for the reactions using double concentration of lactic acid as used in the experiments given in table 1st were determined and are noted in the following table.

TABLE 2

Concentration of lactic acid – 0·376 M
Concentration of potassium dichromate - 0·0025N
Temperature –30°C

Concentration of Sulphuric acid	Average value of K (first order)	Order with respect to lactic acid
Nil	$21{\cdot}77 \times 10^{-3}$	1·4
0·02N	$26{\cdot}77 \times 10^{-3}$	1·1
0·05N	$31{\cdot}66 \times 10^{-3}$	0·95
0·1N	$39{\cdot}25 \times 10^{-3}$	1·0

Thus the total order of the reaction when there was no sulphuric acid present was found to be 2·4 and with increasing concentration of sulphuric acid this value tends to become two.

Temperature coefficient and Heat of Activation

The temperature coefficient for 10°C rise of temperature was calculated and it was found to be 1·75 in the case where no acid was used and 1·6 in presence of acid.

The heat of activations for this reaction in the presence of different concentration of sulphuric acid have been calculated. These values for different temperature ranges have been noted.

TABLE 3

Concentration of sulphuric acid reaction mixture	Temperature range	Heat of activation
Nil	30 – 40°C	10490 calories
Nil	40 – 50°C	9187 ,,
·02N	30 – 40°C	8722 ,,
0·02N	40 – 50°C	8822 ,,
·05N	30 – 40°C	8117 %
·05N	40 – 50°C	8896 ,,
0·1N	30 – 40°C	9721 ,,
0·1N	40 – 50°C	9356 ,,

Determination of equivalents

It should be noted here that the reaction mixtures containing excess of lactic acid in the presence of sulphuric acid upto the concentration of 0·05N gave the test of acetaldehyde after the complete reduction of Hexavalent chromium at 40°C or below. The equivalents of oxidant consumed for lactic acid under similar conditions but containing large excess of chromic acid were determined. The values are given below :

TABLE 4

Lactic acid without sulphuric acid	Equivalents of Cr^6 per gm. mol of lactic acid	Lactic acid with sulphuric acid	Equivalent of Cr^6 per gm. mol of lactic acid
Temperature		Temperature	
30°C	2·5	30°C	4·0
40°C	4·0	40°C	4·0
50°C	4·0	50°C	4·0

From the above data it is seen that when the hydrogen ion concentration is low (No sulphuric acid added to the reaction mixture), the majority of the oxidised product is acetaldehyde and when the oxidation is complete, acetic acid is produced at higher temperature and in the presence of sulphuric acid.

Mechanism

The oxidation of lactic acid may occur by the attack of an oxidising agent at either the carboxylic group or the alcoholic group. Thus :

$$CH_3CHOH\ COOH \xrightarrow{O} CH_3\ CHO + CO_2 + H_2O \qquad (1)$$

$$CH_3CHOH\ COOH \xrightarrow{O} CH_3CO\ COOH + H_2O \qquad (2)$$

In the latter case the product is pyruvic acid, which I have noted to be easily oxidised by hexavalent chromium forming acetic acid even in the absence of sulphuric acid.

$$CH_3CO\ COOH \longrightarrow CH_3COOH + CO_2 \qquad (3)$$

Hence acetaldehyde will appear as a product only if the oxidation takes place at the carboxylic group as in equation (1). I have also noted that the oxidation of acetaldehyde by potassium dichromate in the absence of 'sulphuric acid' is hardly appreciable and the reaction has only a measurable velocity when the sulphuric acid concentration is raised to 0·1N.

In view of these observations it is clear that when the sulphuric acid concentration and the temperature is low the main reaction is the oxidation occurring at the carboxylic group yielding acetaldehyde. It appears that when either the temperature or the concentration of sulphuric acid is raised the acetaldehyde formed may be slightly oxidised to acetic acid. I have repeatedly observed that even at temperature 30°C in the presence of sulphuric acid above 0·05N, the product of oxidation does not contain any acetaldehyde. At this temperature and acid concentration oxidation of acetaldehyde by hexavalent chromium is

however very slow. It is therefore, to be concluded that above 0·05N concentration of sulphuric acid the oxidation of lactic acid as shown in equation (2) also takes place in appreciable amounts. Thus pyruvic acid becomes an intermediatory product which is immediately oxidised to acetic acid.

The oxidation of lactic acid occurring at the alcoholic group may be due to an estar formation with chromic acid as postulated by Westheimer *et al.* for the oxidation of alcohols, but it appears that esterification is only possible when the concentration of sulphuric acid is slightly high. The oxidation of the COOH group may be ascribed to the enolisation of the ketonic group present and subsequent esterification.

It will be, however, noted that heat of activation is sufficiently low for this reaction when compared with such values for several bimolecular reactions. This fact therefore, suggests that the first process may be one electron transfer reaction yielding a free radical. Thus :

$$CH_3\ CHOH\ COOH + HCrO_4^{-1} = CH_3\ CHOH\ COOH^+ + HCrO_4^{-2} \qquad (4)$$

which again reacts with $HCrO_4^{-1}$ and yield acetaldehyde

$$CH_3CHOH\ COOH^+ + HCrO_4^{-1} = CH_3CHO + CO_2 + 2H^+ + HCrO_4^{-2} \qquad (5)$$

The pentavalent chromium will react with lactic acid in the same way as the hexavalent chromium and will form quadrivalent chromium which may subsequently decompose into pentavalent and trivalent chromium :

$$2CrO_2 + H^+ \longrightarrow HCrO_4^{-2} + Cr^3 \qquad (6)$$

It is assumed that the reaction (4) is the rate determining step then the order of the reaction should be unity with respect to lactic acid and one with respect to oxidant. It appears that at the higher concentration of sulphuric acid and at higher temperature the oxidation occurs at hydroxyl group forming as ester with the anion of hexavalent chromium.

$$HCrO_4^{-1} + 2H^+ + CH_3CHOH\ COOH \rightleftharpoons CH_3 - \underset{\displaystyle COOH}{\overset{\displaystyle H}{\overset{|}{\underset{|}{C}}}} - O - CrO_2 - OH_2^+ + H_2O$$

which is followed by :

$$CH_3CO\ COOH + 2H_2O + H^+ + CrO_2$$

The oxidation which forms pyruvic acid requires two equivalents of oxidants and yields no acetaldehyde. My results also show that four equivalents of the oxidant are consumed at higher temperature and at higher concentration of sulphuric acid with the formation of acetic acid.

The function of sulphuric acid in the reaction may now be discussed. My experimental data on the oxidation of lactic acid and lactate mixture show that as the concentration of lactate is increased the reaction velocity sharply decreases and is attended with an induction period. This suggests that the reaction (4) with hexavalent chromium is with the undissociated lactic acid. It is clear that the intermediatory radical $CH_3CHOH\ COOH^+$ can be possible when the tendency of the removal of the proton from lactic acid is considerably, checked by the increase in hydrogen ion concentration.

It should be also noted here that Mn^{+2} or manganous sulphate greatly accelerates the oxidation of lactic acid and this behaviour is different from that observed for the oxidation of alcohols by chromic acid. This happens because of the tendency of the lactic acid to form a complex compound with trivalent manganese. I have repeatedly observed that especially at low hydrogen ion concentration a mixture of chromic acid lactic acid and manganous sulphate yields an orange red colour solution which gradually fades off, a phenomenon which is similar to that observed by Chakravarty and Ghosh for the oxidation of oxalic acid by chromic acid catalysed by bivalent manganese.

My experimental results reported here therefore lead to the conclusion that the lactic acid containing an alcoholic group and a carboxylic group is attacked by the oxidant at both the places depending upon the acid range of the medium and temperature. The product of the oxidation may be aldehyde, acetic acid or their mixture according to the conditions of the experiments.

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References

1. Chapman and Smith. *J. Chem. Soc.*, **20**, 173, 1967.
2. Dhar and Dey. *Z. Electro Chim.*, **32**, 586, 1926.
3. Dhar and Bhattacharya. *Ann. Chim.*, (9), **19**, 130, 1919 ; *J. Chem. Soc.*, **111**, 707, 1917.
4. Westheimer. *Chem. Rev.*, **45**, 419, 1949.
5. Chatterji and Antony. *Proc. Nat. Acad. Sci. India*, **23A**, 20, 1954 ; *Zeit Phys. Chem.*, **210**, 50, 103, 1959.

5*a*. Chatterji, *et al.*, *ibid*, **228**, 173, 1965.

6. Westheimer and Novick. *J. Chem. Phys.*, **11**, 500, 1943.

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Inversion of a Convolution Transform with Kelvin's function as Kernel

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Abstract

In this paper author has inverted a convolution transform whose Kernel is a Kelvin's function, by using the operational Calculus.

Introduction

Inversion intergrals are known for the integral transformations which involve Chebyshev polynomial[5], Legendre's polynomial[1], Gengenbaur polynomial[2], or Simple Laguerre polynomial[4], as Kernel. In the present paper we have inverted a convolution transform whose Kernel is a Kelvin's function. The method of Laplace transform has been used.

The Kelvin's functions, Bessel real and Bessel imaginary, are of great practical significance in the study of current density in a wire carrying alternating currents, skin effect in circular tubes, eddy current loss in a core of solenoid etc. (See, Bessel functions for engineers, N. W. McLachlan, Clarendon Press, Oxford Second edition 1955). Kelvin's function *ber* (t) and *bei* (t) are defined as

$$ber\,(t) = \tfrac{1}{2}\,[\,J_0\,(te^{\frac{3}{4}\pi i}\,) + J_0\,(te^{-\frac{3}{4}\pi i})] \tag{1·1}$$

and

$$bei\,(t) = \tfrac{1}{2i}\,[\,J_0(te^{\frac{3}{4}\pi i}) - J_0\,(te^{-\frac{3}{4}\pi i})]. \tag{1·2}$$

Laplace Transform of a function $f(t)$ is defined as the integral

$$\int_0^\infty e^{-pt} f(t)\,dt = F(p).\quad Re(p) > 0. \tag{1·3}$$

For brevity, this will be denoted symbolically as

$$f(t) \doteqdot F(p). \tag{1·4}$$

If $f(t) \doteqdot F(p)$ then

$$D^n f(t) \doteqdot P^n f(p) - p^{n-1} f(0) - p^{n-2} f'(0) \ldots - f^{n-1}(0) \tag{1·5}$$

The following known results will be used in the sequel.

$$\sqrt{2}\ ber\,(at) \doteqdot \frac{[(p^4 + a^4)^{\frac{1}{2}} + p^2]^{\frac{1}{2}}}{(p^4+a^4)^{\frac{1}{2}}} \tag{1·6}$$

$$\sqrt{2}\ bei\ (at) \doteqdot \frac{[(p^4 + a^4) - p^2]^{\frac{1}{2}}}{(p^4+a^4)^{\frac{1}{2}}} \tag{1·7}$$

(1·8) $\int_0^t f_1(u) f_2(t-u) \doteqdot g_1(p) g_2(p)$, where

$f_1(t) \doteqdot g_1(p)$, and $f_2(t) \doteqdot g_2(p)$.

2. *Statement*—If $f(t)$ and its derivatives $f^m(t)$ are sectionally continuous for $0 \leqslant x < x_1 < \infty$ $0 \leqslant m \leqslant 5$ and $f^r(0) = 0$, for $0 \leqslant r \leqslant 4$ then

(2·1) $$\int_0^\infty \sqrt{2}\ ber\{a(x-t)\}\, g(t)\, dt = f(x)$$

has for the unknown function $g(x)$, the solution

(2·2) $$a^2 g(x) = \int_0^x \sqrt{2}\ bei\{a(x-t)\}\, (D^4 + a^4) f(t)\, dt \text{ for } 0 \leqslant x < x_1$$

Proof--Let $g(t) \doteqdot G(p)$, $f(t) \doteqdot F(p)$.

Applying (1·5) under the condition $f^r(0) = 0$, $0 \leqslant r \leqslant 4$, we get

(2·3) $$(D^4 + a^4) f(t) \doteqdot (p^4 + a^4) F(p).$$

Taking Laplace transform of the equation (2·1) and applying (1·8), we get

$$F(p) = \frac{[(p^4 + a^4)^{\frac{1}{2}} + p^2]^{\frac{1}{2}}}{(p^4 + a^4)^{\frac{1}{2}}}\ G(p)$$

Therefore

$$G(p) = \frac{(p^4 + a^4)^{\frac{1}{2}}}{[(p^4 + a^4)^{\frac{1}{2}} + p^2]^{\frac{1}{2}}} F(p).$$

A little adjustment gives,

(2·4) $$G(p) = \frac{[(p^4 + a^4)^{\frac{1}{2}} - p^2]^{\frac{1}{2}} (p^4 + a^4)}{(p^4 + a^4)^{\frac{1}{2}} a^2} F(p)$$

Taking the inverse Laplace transform of the equation (2·4) in the light of (1·7) and (2·3) we get (2·2). This concludes the proof.

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References

1. Buschman, R. G. An Inversion Integral for a Legendre transformation. *American Math. Monthly*, **69**, 288-289, 1962.
2. Buschman R. G. An Inversion Integral. *Proc. Amer. Math. Soc.*, **13**, 675-677, 1962.
3. Erdely, A. *et al.* Tables of Integral transforms. *McGraw Hill, New York*, Vol. 1, 1954.
4. Widder, D. V. The inversion of a convolution transform whose Kernel is a Laguerre polynomial. *American Math. Monthly*, **70**, 291–295, 1963.
5. Tali. A new class of Integral transforms. *Proc. American Math. Soc.*, **11**, 290-298, 1960.

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Effect of Nitrogenous Fertilisers on the yield of Paddy and on Soil in different regions of West Bengal†

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It has been reported by Sethi *et al*[1] that paddy crop gives a positive response to the application of nitrogen in almost all types of Indian soils. Bartholomew[2] found that continuous application of ammonium sulphate year after year decreased the soil pH and produced deleterious effect on the crop yield. Chang, Lin and Puh[3] observed that continuous application of ammonium sulphate for a period of 24 years on an acid soil in Taiwan gave in general a good yield for the first few years but afterwards there was a loss of soil fertility and decrease in crop yield.

Anderson, Jones and Armiger[4] observed that on aman paddy, ammonium sulphate acted as the most efficient nitrogen fertiliser and urea preferable in comparison to ammonium nitrate. On applying 20 to 30 lb. N per acre on paddy, Ramiah, Ghose and Vachhani[5] observed the relative efficiency of the different forms of nitrogenous fertilisers as Amm. Sulph. > Amm. Nitrate > Amm. Phosphate > Urea > Cal. Cyanamide > Pot. Nitrate > Sodi. Nitrate.

Sethi[6] stated that the most suitable time of application of nitrogenous fertilisers to paddy plants varied at different places ; at some places the application at different ages of the growth of the plant did not show any effect, whilst at others a marked difference was observed. Again at some places total quantity applied at one time proved better than its application in split doses at two or three times. Pearsall[7] observed that when fertilisers like ammonium sulphate was applied on the soil surface or oxidation layer, the ammonia was converted to nitrate which passes into lower reducing zone and was reduced to nitrite and then to elementary nitrogen which got lost. Abichandani and Patnaik[8] found that sub-surface placement increased availability of nitrogen and decreased fertiliser losses in surface drained waters. These authors[9] further observed that the efficiency of sub-surface placement for increasing grain yield to be about 2·36 times that of surface application.

In view of the foregoing observations cited, to study the effect of nitrogenous fertilisers on the yield of paddy and soil conditions in West Bengal, a large number of field experiments on aman paddy (indica local variety) by the application of different doses and form of nitrogenous fertilisers have been carried out in different soil climatic regions of West Bengal and the results of some of the experiments have been presented in this paper. Time, method of application and residual effects of nitrogenous fertilisers on the yield of paddy and on soil have also been studied and discussed.

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Experimental

The experiments were conducted on transplanted aman paddy in Government Agricultural Farms of West Bengal situated at different soil climatic zones. The land was prepared with the break of monsoon in June and in early August the soil was puddled and transplanting was done at the rate of two seedlings per hole in spacing 9″ × 9″. These experiments were continued for several years without changing the site, and the layout. The crop was grown normally under rainfed condition and harvested when mature by the middle of December each year. Chemical analysis of the soil of the experimental plots were conducted from time to time to find out the effect of fertilisers on soil composition. The grain yields in quintals/hectare obtained are as follows :

Results and Discussion

(*A*) *Effect of continuous application of ammonium sulphate at different doses* :

Location	Ammonium sulphate (doses of N in kg/ha)				
1. Chinsurah (Gangetic alluvial neutral clay soil) (Average of 10 years) :	Control	33·6	67·2	100·8	134·4
	20·9	25·3	25·1	24·9	23·1
	'F' test-Significant at 1% level; S. E. 0·589 C.D. at 5% level : 1·72 ; C.D. at 1% level : 2·33				
2. Sriniketan (Lateritic sandy loam) (Average of 6 years) :	Control	33·6	67·2		
	15·6	20·6	21·1		
	'F' test-Significant at 1% level ; S. E : 0·241 C.D. at 5% level : 0·69 ; C.D. at 1% level : 0·94				
3. Canning (Saline soil) (Average of 3 years)	Control	11·2	22·4	33·6	
	15·4	18·7	20·6	24·4	
	'F' test-Significant at 1% level ; S. E. : 0·57 C.D. at 5% level : 1·62 ; C.D. at 1% level : 2·17				

The results show that the optimum response of paddy to nitrogen has been obtained with 33·6 kg N/ha. In alluvial clay region, with the increase in dose of nitrogen beyond 33·6 kg, there is a deterioration in the grain yield. In the lateritic sandy region, however, we can go up to 67·2 kg. of nitrogen/ha, but the yield obtained is not significantly different than that obtained with 33·6 kg of nitrogen/ha.

Chemical analysis of top soil (0″-6″ depth) under different treatments done before and after the experiment both at Chinsurah and Sriniketan are given here.

Mechanical and chemical analysis of soil of the experimental site before the start of the experiment (on oven dry basis)

	Coarse sand %	Fine sand %	Silt %	Clay %	Nitrogen %	Organic Carbon %	Ex–Ca in m.e. %	Ex. bases in m.e. %	pH
Chinsurah	0·33	7·20	32·66	55·00	0·083	0·755	18·14	–	6·6
Sriniketan	35·73	35·06	15·83	13·38	0·038	0·034	3·40	4·08	6·0

Chemical analysis of soil of the experimental site at the end of the experiment (on oven dry basis)

Treatments (per hectare)	Nitrogen (per cent)	Organic Carbon (per cent)	Ex. calcium m.e. (per cent)	Ex. bases m.e. (per cent)	pH
		CHINSURAH			
0 kg N	0·083	0·755	18·12	–	6·6
33·6 kg N	0·081	0·712	18·10	–	6·6
67·2 kg N	0·083	0·739	17·96	–	6·5
100·8 kg N	0·080	0·769	18·00	–	6·4
134·4 kg N	0·082	0·722	17·98	–	6·4
		SRINIKETAN			
0 kg N	0·038	0·330	3·40	4·08	6·0
33·6 kg N	0·038	0·310	2·75	3·88	6·1
67·2 kg N	0·038	0·300	3·00	3·88	6·1

It will be seen from the foregoing results that at Chinsurah neutral clay soil there has not been any appreciable change in soil composition due to continuous application of ammonium sulphate year after year even after 10 years of experiment. There is, however, some lowering of pH due to application of 100·8 kg and 134·4 kg N/hectare but the change is not very significant. At Sriniketan lateritic sandy loam soil however, there is some decrease in organic carbon, exchangeable calcium and exchangeable bases due to continuous application of ammonium sulphate after 6 years of the experiment but the pH of the soil has practically remained constant. These changes, however, are not marked. Chemical analysis of soil at Canning (saline soil) has also been done after the end of the experiment but no appreciable change in the chemical composition of the soil has been observed here also.

It is generally said that soils get hard due to continuous application of nitrogenous fertilisers. Penetrometer studies were, therefore, taken at Chinsurah after 10th year of the continuous application of ammonium sulphate.

Following results have been obtained :

Treatments per hectare

	0 kg N	33·6 kg N	67·2 kg N	100·8 kg N	134·4 kg N
Penetration in inches	2·80	2·80	2·78	2·76	2·70
Moisture %	14·00	13·78	13·05	13·01	12·90

The results show that soil has become slightly hard and has lost moisture to some extent due to continued application of 100·8 kg and 134·4 kg N/hectare. No appreciable change has, however, been observed by the application of 33·6 kg and 67·2 kg N/hectare.

It can, therefore, be concluded that ammonium sulphate in low doses (20 to 45 kg N/ha) may be applied continuously for some years in different soil regions of West Bengal for increasing the yield of paddy without any deleterious effect on soil.

(*B*) *Comparison of different forms of nitrogenous fertilisers* :

The effect of different forms of nitrogenous fertilisers on the yield of paddy may be seen from the following results obtained in different districts :

Nitrogenous fertilisers (doses of N in kg/hectare)

Location	Control No N	Ammo. Sulphate	Ammo. Chloride	Sodi. Nitrate					
1. Burdwan (Damodar Flat land clay loam)		22·4	22·4	22·4					
(Average of 2 years	19·4	24·3	24·4	25·3					
2. Bankura (Laterite		Nitro. Phos.	Ammo. Phos.	Cal. Ammo. Nitrate					
sandy loam)		33·6	33·6	33·6					
(Average of 2 years)	16·7	24·9	27·1	25·3					
3. Burdwan (Damodar		Ammo. Sulphate		Ammo. Nitrate			Urea		
Flat land clay loam)		16·8 : 33·6 : 50·4		16·8 : 33·6 : 50·4			16·8 : 33·6 : 50·4		
(Average of 3 years)	24·3	27·5	31·0	33·5	27·3	29·2	31·3	27·5	30·9	32·9
4. Chinsurah (Gangetic alluvial neutral clay)										
(Average of 3 years)	18·3	19·2	20·5	20·9	19·5	19·4	19·0	19·8	21·3	19·8
5. 24-Parganas (Farmers'		Ammo. Sulphate			Ammo. Nitrate					
plot saline soil)		22·4 : 44·8			22·4 : 44·8					
(Average of 2 years)	13·6	17·1 : 20·3			17·1 : 20·6					
6. 24-Parganas (Farmers'		Control	Ammo. Sulphate		Urea					
plot saline soil)		No N	22·4 : 44·8		22·4 : 44·8					
(Average of 2 years)		15·9	20·7	24·4	20·6	22·9				
7. Mainaguri	Control	Amm. Sulph.	Amm. Sulph. Nitr.		Urea		Cal. Amm. Nitr.			
(sandy loam)	No N	22·4 : 44·8	22·4 : 44·8		22·4 : 44·8		22·4 : 44·8			
(Average of 3 years)	17·6	22·6 : 22·9	21·6 : 23·1		23·0 : 22·9		21·5 : 23·5			
8. Coochbehar (Loam and clay loam) (Average of 3 years) :	14·9	17·3	16·9	15·9	17·1	16·1	16·9	15·8	15·3	

From the foregoing results it will be seen that ammonium sulphate appears to be slightly better than other nitrogenous fertilisers in grain production. Urea and ammonium chloride can be compared favourably with ammonium sulphate and have proved superior to ammonium sulphate nitrate, calcium ammonium nitrate, ammonium nitrate etc. There is, however, no significant difference between these fertilisers amongst themselves in grain producing power.

Since urea costs less per unit of nitrogen and has a lower potential for production of unsaturation in the soil complex compared to ammonium sulphate

it seems to be more suitable for West Bengal soils. A majority of West Bengal soils are acidic and have a tendency to remain so due to climatic factors, hence, a nitrogenous fertiliser having low potential acidity like urea will be better in the long run, instead of other acid producing fertilisers.

(*C*) *Effect of different times of application of nitrogenous fertilisers* :

Ammonium sulphate and Urea at the rate 33·6 kg N/ha each were applied at 5 different time *i.e.* (T_1) full dose at puddling, (T_2) full dose 4 weeks after transplantation, (T_3) ½ dose at puddling and ½ dose at 4 weeks after transplantation (T_4) 1/3 dose at puddling, 1/3 dose at 4 weeks after transplantation and 1/3 dose at flowering and (T_5) ½ dose at 4 weeks after transplantation and ½ dose before flowering. The results obtained are as follows :

Location : Chinsurah (Gangetic alluvial neutral clay)

(Average of 5 years)

Fertiliser (F) in kg/hactare	Times of application (T) T_1	T_2	T_3	T_4	T_5	Average (F)
1. Ammonium Sulphate	20·9	20·6	21·1	20·8	21·2	20·9
2. Urea	19·1	19·3	19·8	18·1	18·8	19 8
Average (T) :	20·0	19·9	20·4	19·4	20·0	

'F' test for (F)—Significant at 1% level
'F' test for (T)—Not significant at 5% level
S. E. of difference of fertiliser means : 39·9
C.D. for (F) at 5% level : 85·2 ; at 1% level : 117·8

It can be seen from these results that application of fertilisers with ½ dose at puddling and ½ dose at 4 weeks after tran plantation is to some extent better than other times of application followed here, the difference, however, is not significant Amongst two types of nitrogenous fertiliser, grain yield obtained with ammonium sulphate is significantly higher than that obtained with urea.

Results of chemical analysis of soil sample done before and those of five composite soil samples corresponding to five different times of application of both ammonium sulphate and urea after the end of the experiment are as follows :

Chemical analysis of the soil of experimental site before the start of the experiment (on oven dry basis)

pH	Total N%	Total C%	Av. P_2O_5	Ex. Ca(m.e%)	Ex. bases(m.e%)
6 4	0·082	0·796	0·0054	15 4	21·5

Chemical analysis of the soil of experimental site at the end of the experiment (on oven dry basis)

	T_1	T_2	T_3	T_4	T_5
pH	6·2	6·4	6·1	6·4	6·3
Total N%	0·086	0·084	0·083	0·085	0·091
Total C%	0·782	0·786	0·790	0 802	0·814
Av. P_2O_5	0·0061	0·0056	0·0063	0·0064	0·0061
Ex. Ca(m.e.%)	14·8	14·6	15·2	15·1	15·6
Ex. Bases(m.e.%)	20·8	21·2	20·5	19·8	19·6

It is clear from the foregoing results that there is no appreciable change in the chemical composition of the soil due to the application of small doses of nitrogenous fertiliser continuously for five years even when they are applied at different times of the growth of the plant in split doses.

It can, therefore, be concluded that times of application of nittogenous fertilisers to paddy plants have not much to commend on the specific date under water logged condition of West Bengal. The favourable time of application seems to be during puddling or one month after transplantation. It is, however, always better to apply nitrogenous fertilisers in split doses where there is a tendency of lodging.

(*D*) *Effect of placement of nitrogenous fertilisers* :

In these experiments ammonium sulphate was mixed up with soil and formed into small pellets. These pellets were placed 3 inches below the surface of soil one month after transplantation of paddy. Following results have been obtained :

Doses of N in kg/ha	BURDWAN (Damodar Flat land clay loam)			CHINSURAH (Gangetic alluvial neutral clay)		
	Surface application (1)	Placed (2)	Increase over surface application (2)—(1)	Surface application (1)	Placed (2)	Increase over surface application (2)—(1)
22·4	32·5	34·2	+1·7	30·8	29·9	-0·9
44·8	34·2	35·6	+1·4	30·4	31·0	+0·6
67·2	36·2	36·3	+0·1	27·1	28·2	+1·1

The results indicate that increase in grain yield due to placement of ammonium sulphate below soil surface over surface application is not significant. De and Digar[9] in West Bengal found that placement of ammonium sulphate 3 inches below soil surface reduced loss of nitrogen to some extent but even then it was fairly high. From this it is evident that in cropped paddy field the oxidizing zone

extends fairly deep below the soil surface. Placing the fertiliser further below will probably deprive shallow root system of paddy plants to absorb the applied nutrient. Hence, it appears doubtful whether placement of nitrogenous fertilisers below soil surface in water logged paddy field is of much benefit. However, it is very necessary to undertake further experiments in this line under various soil conditions.

(*E*) *Residual effect of nitrogenous fertilisers* :

In this experiment ammonium sulphate was applied continuously for 5 years at rate of 33·6 kg N and 67 2 kg N per hectare. In the 6th year the application of ammonium sulphate was stopped and residual effects of the treatments were studied. Following results have been obtained :

Location	Ammonium Sulphate		
	Control (0 kg N) (1)	33·6 kg N (2)	67·2 kg N (3)
Sriniketan (Lateretic sandy loam)	9·2	8·4 (2-1)	7·7 (3–1)
	Residual effect	-0·8	-1·5

'F' test for N - Non significant

The results indicate that application of nitrogen as ammonium sulphate produces a negative non-significant residual effect on grain yield.

Summary

The applications of different doses, forms, time, method and residual effects of nitrogenous fertilisers on the yield of paddy (indica local variety) and on soil have been studied in different soil climatic zones of West Bengal.

It has been reported that ammonium sulphate in low doses (20 to 45 kg N/ha) may be applied continuously for some years in the same plot of land in different regions of West Bengal for increasing the yield of paddy without any deletarious effect on soil.

Effect of ammonium sulphate appears to be slightly better than other nitrogenous fertilisers in grain production. Urea and ammonium chloride can be compared favourably with ammonium sulphate and have proved superior to ammonium sulphate nitrate, calcium ammoninm nitrate, ammonium nitrate etc. Since urea costs less per unit of nitrogen and has a lower potential for production of acidity and unsaturation in the soil complex compared to ammonium sulphate it seems to be more suitable for West Bengal soils in the long run.

Times of application of nitrogenous fertilisers to paddy plants have not much to commend on the specific date under water logged condition of West Bengal. The favourable time of application seems to be during puddling or one month after transplantation. Split application of nitrogenous fertilisersis however, always better where there is a tendency of lodging. It also appears doubtful whether placement of nitrogenous fertilisers below soil surface in water logged paddy fields is economic and is of much benefit. Further experiments in this line under various soils conditions of West Bengal is, however, necessary.

Application of nitrogen as ammonium sulphate has produced a negative non-significant residual effect on grain yield in lateritic sandy loam soil occurring in some parts of West Bengal.

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References

1. Sethi, R. L., Ramiah, K. and Abraham, T. P. *Bull. Indian Coun. Agric. Res.*, **38**, 1952.
2. Bartholomew, R. P. *Jour. Amer. Soc. Agron.*, **20**, 1303, 1928.
3. Chang, S. C., Lin, J. F. and Puh, Y. S. *Bull. Agric. Res. Inst. Tech. Taiwan*, **3**, 18, 1947.
4. Anderson, M. S., Jones, J. W. and Armigar, W. H. *Jour. Amer. Soc. Agron.* **38**, 743-753, 1946.
5. Ramiah, K., Ghosh, R. L. M. and Vachham, M. V. *Emp. J. Expt. Agric.* **20**, 161-174, 1952.
6. Sethi, R. L. *Mem. Dept. Agric. Ind. Bull.*, **11**, 87, 1936.
7. Pearsall, W. H. *Emp. J. Agric.*, **18**, 289, 1950.
8. Abichandani, C. T. and Patnaik, S. *Jour. Ind. Soc. Soil Sci.*, **6**, 87, 1958.
9. Abichandani, C. T. and Patnaik, S. *Jour. Ind. Soc. Sci.*, **7**, 171, 1959.
10. De, P. K. and Digar, S. *Jour. Agric. Sci.*, **45**, 3, 1955.

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The studies on the distribution of activity among the different carbon atom of aminoacids formed from the photochemical oxidation of labelled glucose (- ^{14}CHO)

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Abstract

It has been observed that different labelled aminoacids are formed during the photochemical oxidation of labelled glucose CH_2OH $(CHOH)_4$ ^{14}CHO. From the detailed analyses for the activity distribution of isotopic carbon (C^{14}) of labelled glucose it has been found that the activity of the labelled aldehydic carbon atom of glucose is distributed among the first and second carbon atoms in the amino acid molecule.

Although the use of radioactive isotopes has been extensively reported in recent years, it is by no means a new technique. Radioactive tracer techniques are of very wide application and have been utilized considerably in medicine and biology, but in chemistry, physics and metallurgy they have not yet been so extensively applied. Nevertheless, many of the problems which arise in these sciences can be studied by radiotracer methods and considerable advances are likely by their use in the future. The technique has most often been applied using a radioactive isotope as a tracer for a stable isotope both present together.

Many scientific workers like Loeb[17], Bernal[5], Stanley and Miller[18] studied the formation of amino acids from nitrogenous compounds under the influence of electric discharges. Dhar[10] has been able to synthesise in glass vessels exposed to tropical sunlight, several organic and inorganic compounds which have been obtained by Baly[2] and his co-workers, using quartz vessels exposed to ultraviolet from mercury lamp, and is of the opinion that ordinary chemical reactions may be sensitive to several radiations, the effects of which are additive. Dhar and Mukherjee[11] obtained aminoacids by the action of nitrates on carbohydrates in presence of light using titania as photocatalyst. Oparin[20] and Pavalovskaya *et al*[22] are of the opinion that ultraviolet radiation from the sun is the only responsible factor for photosynthesis of aminoacids on the earth. Recently Oro[21] has also been successful in synthesising aminoacids together with organic compounds with the help of electric discharge from a tungsten lamp. Banerjee and Dhar[3] investigated the formation of aminoacids from the photochemical oxidation of glucose on metallic oxides surface in presence of light under sterile condition.

Von Euler, Lindberg[12] and Bernoulli and Cantieni[6] noted the formation of acid, carbon monoxide, carbon dioxide and hydrogen when D-glucose solutions

were irradiated by ultraviolet rays. The decomposition is accelerated by acids and retarded by alkali. Laurent and Wertheim had used strong emission lines, with the order of intesity of 2540, 2650, 2970, 3020, 3130, 3650 Å and they found that most active part of the light is below 2800 Å for the decomposition of glucose solutions.

During irradiation, carbondioxide is evolved continuously. The rates of acid formation and of carbon dioxide evolution increase with the time-period of irradiation and with increasing concentration of D-glucose. This indicates that direct absorption of radiation and not indirect action, is the dominant process responsible for the degradation. The concentration of D-glucose falls linearly with the irradiation time and the products accumulated, in turn, to maximum concentration and are thereafter destroyed. If irradiation is continued upto 160 hours paper chromatographic examination indicates that no detectable amounts of the products remain and the initial D-glucose has been converted completely to carbon dioxide.

Photofixation of nitrogen was first studied by Dhar and Mukherjee[11]. Turner[25], Sulaiman[24] and many others confirmed Dhar's theory of nitrogen fixation. Carbohydrates undergo slow oxidation in presence of light and water according to the following equations :

$$C_6H_{12}O_6 + 6O_2 + h\nu = 6CO_2 + 6H_2O + 676 \text{ K. cals.} \quad (1)$$

$$C_6H_{12}O_6 + h\nu = \text{different aliphatic acids} \quad (2)$$

$$C_6H_{12}O_6 + h\nu = CH_4 + CO_2 + CO + H_2 \quad (3)$$

Von Euler and Lindberg[12], Bernoulli and Cantieni[3] have studied the formation of aliphatic acids (monobasic and dibasic acids) from the oxidation glucose. The aliphatic acids thus formed undergo photochemical decomposition and give methane and higher members of this group.

$$CH_3COOH + h\nu = CH_3COO + H \quad (4)$$

$$H + CH_3COOH = H_2 + CH\,COO \quad (5)$$

$$H + CH_3COOH = CH_4 + COOH \quad (6)$$

$$H + CH_3COO = CH_4 + CO_2 \quad (7)$$

$$COOH = CO + OH \quad (8)$$

$$2CH_3COO = C_2H_6 + 2CO_2 \quad (9)$$

$$CH_4 + CO = CH_3CHO \quad (10)$$

The energy released as shown in the equation (1) is utilised in decomposing water into H and OH radical.

$$H_2O + 112 \text{ K. cals} = H + OH \quad (11)$$

Under the radiation of shorter wave length (< 3200 Å) water decomposes photochemically into hydrogen atoms and neutral hydroxyl radicals in presence of zinc oxide and iron oxide etc. as photosensitizer[14,23,29,15]

$$H - OH + h\nu = H + OH \quad (12)$$

It has been studied by Compton, Langmuir[9] and others that normal nitrogen molecules become active and partially ionized due to ultraviolet radiation of wave length between 2900 to 3200 Å. The atomic hydrogen and partially ionized nitrogen reacts and forms ammonia

$$6\dot{H} + N_2^+ = 2NH_3 \quad (13)$$

Banerjee[4] repeated the experiment of Wöhler[28], Fisher, Peters[13], Briner[8] and found that under the influence of ultraviolet radiation ($< 3000\,\mathring{A}$) carbon monoxide, hydrogen and nitrogen forms hydrocynic acid. In the later stage hydrocynic acid and ammonia in presence of water reacts with aldehyde (cf equation 10) and give amino acids.

$$RCHO + HCN + NH_3 = RCH(NH_2)CN + H_2O \quad (14)$$

$$RCH(NH_2)CN + 2H_2O = RCH(NH_2)COOH + NH_3 \quad (15)$$

The present communication deals with the study of the formation of active amino acids from photochemical oxidation of labelled glucose ($-C^{14}HO$). The idea of using the labelled glucose aldehydic carbon atom active, is to ascertain the position of isotopic carbon atom in the amino acid molecule and the distribution of the activity of the aldehydic carbon atom of glucose among the carbon atoms of amino acids form in the system.

Experimental

In ten 250 ml. conical flasks (Pyrex) mixtures of glucose (B. D. H.) and nickel oxide (E. Merck), and glucose, dicalcium phosphate (B. D. H.) were taken separately. Glucose and dicalcium phosphate was added to these systems at the rate of 5% and 0·25% as P_2O_5 respectively of the total content. Labelled glucose $CH_2OH(CHOH)_4{}^{14}CHO$ supplied by Radiochemical centre. Buckinghamshire, England, was added at the rate of 16 μc/g of glucose to each of these flasks. In actual experiments the labelled glucose was mixed at the above rate with the unlabelled glucose solution.

Before weighing, the nickel oxide was washed with double distilled water, dried completely and tested for the absence of nitrogen. Now known quantities of nickel oxide, dicalcium phosphate and the mixture of glucose solution, were taken in ten conical flasks. Distilled water was added to these systems at the rate of 30% of the total content. These flasks were plugged with surgical non-absorbent cotton and sterilised in the Arnold steam sterilizer continuously for three days one hour a day. Afterwards it was tested for any bacterial contaimination by the plate method using Beijerinck's media.

These flasks were arranged in a circular fashion and exposed under the ultraviolet light at a distance of 30 cms from it as shown in the diagram. After 30, 60, 90, 120, 150 hours of exposure samples were analysed for amino acids both qualitatively and quantitatively. For qualitative determination paper chromatographic method was applied and Rf values were determined. Quantitative estimation of amino acids were done by using total colour density technique. For these estimations Photovolt Electric Densitometer (Mod. 525) was used. The accuracy in this method has been found to be of very high order and it varies by $\pm$ 0·01 percent.

The labelled amino acids formed in this oxidation reaction were determined by autoradiograph technique. The circular chromatographs were kept in contact with X-ray film for fifteen days. These negative plates were then developed and printed on photographic printing paper. The pH of the solution was determined by Beckman pH meter.

Determination of activity for individual carbon atom in amino acids

For the study of activity of individual carbon atom in the amino acid molecule, amino acids were separated after 120 hours by column chromatography technique. Each fraction of these amino acids was then subjected to combustion by D. D. Van Slyke and Jordi Folch[20] method. In this method sodium carbonate labelled with C^{14}, was treated with lactic acid and gave sodium lactate and $C^{14}O_2$ gas. The activity of $C^{14}O_2$ formed in this method was measured by Bernstein and Ballentine method. This gave the total activity of the compound.

These amino acids were subjected to decarboxylation with ninhydrin which removed C_1.

$$RCH(NH_2)C^1OOH + \text{ninhydrin} \rightarrow RCHO + NH_3 + C^1O_2$$

For the above reaction aminoacid was boiled with ninhydrin (0·1 gram) and trapped the CO_2 evolved in strong solution of sodium hydroxide. The sodium carbonate labelled with C^{14} formed in this reaction was treated with lactic acid and it gave sodium lactate and $C^{14}O_2$. The activity of $C^{14}O_2$ was measured. The difference of this activity with the total activity gave the activity of C_1 carbon atom. This shows that some part of the active carbon atom of labelled glucose ($-C^{14}HO$) occupied the first carbon atom position in aminoacid. The active glutamic acid was subjected to Schmidt's reaction yielded diaminobutyric acid. Combustion of the latter and measured the activity. This activity was exactly the same as the original sample. This proved that the fifth carbon atom (C_5) was not active. From $\alpha-\gamma$ – diaminobutyric acid, $\alpha-\gamma$ – diaminobutyric acid hydrochloride[16] was prepared. This was subjected to silver oxide oxidation yielded β – alanine. This compound was subjected to combustion and the activity of evolved $C^{14}O_2$ was determined. This gave the activity of C_{2+3+4} atom. It was observed that this activity was equivalent to the activity of the remaining compound after the removal of C_1. β – alanine on fusion with KOH – NaOH[1,19,27] led to the formation of acetic acid. The activity of this compound was determined by combustion method and found that the activity was same as β – alanine, this showed that C_4 does not contain any activity. This means the activity of acetic acid must be with the second or third carbon atom or in both the carbon atoms. Sodium acetate was prepared from the acetic acid. Fused sodium acetate and powdered sodalime was heated in a copper tube for half an hour whole methane gas was removed by this operation. The carbonate obtained was treated with lactic acid and CO_2 formed, its activity was determined. This proved that the second carbon atom of glutamic acid was also active.

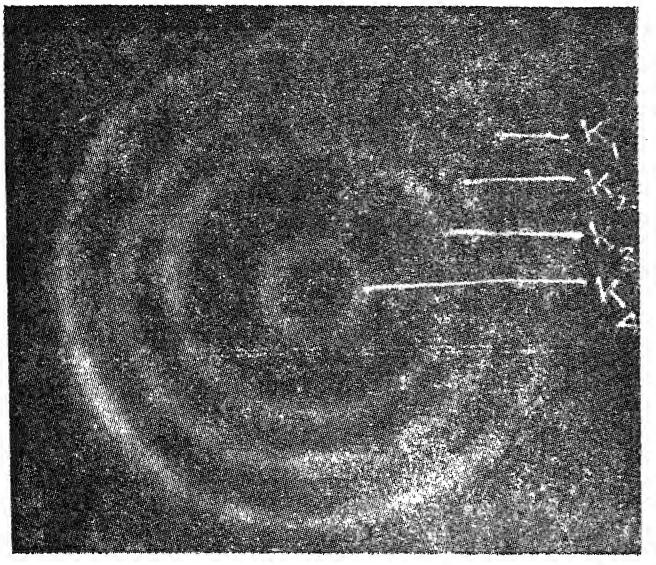
$$HOOC^5\text{-}{}^4CH_2\text{-}C^3H_2\text{-}{}^2CH\text{-}NH_2\text{-}{}^1COOH \xrightarrow[\text{reaction}]{\text{Schmidt's}} {}^4CH_2\text{-}NH_2\text{-}{}^3CH_3\text{-}{}^2CHNH_2\text{-}{}^1COOH + {}^5CO_2$$

$$\downarrow Ag_2O \text{ oxidation}$$

$$H_2O + {}^3CH_3\text{-}{}^2COONa \xleftarrow{NaOH} {}^3CH_3\text{-}{}^2COOH + {}^4CO_2 \xleftarrow[\text{Fusion}]{KOH\text{-}NaOH} {}^4CH_2NH_2\text{-}{}^3CH_2\text{-}{}^2COOH + {}^1CO_2$$

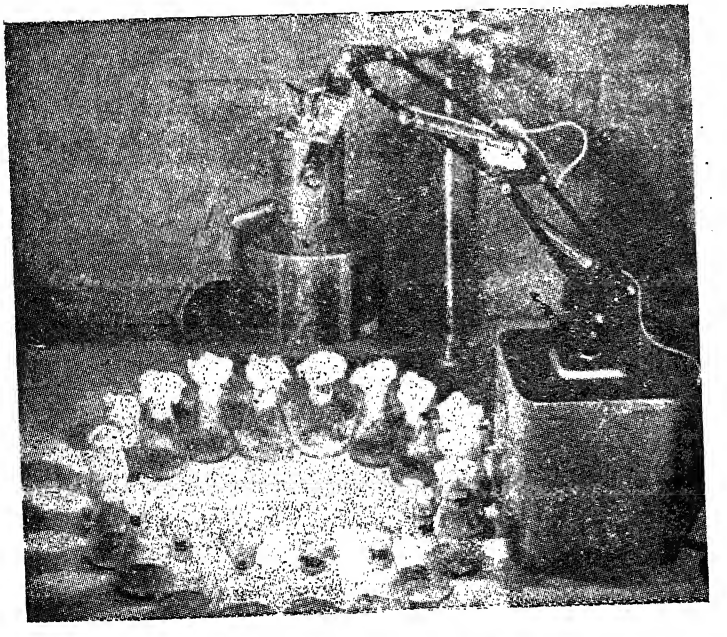
$$\downarrow NaOH$$

$${}^3CH_4 + Na_2\,{}^2CO_3$$



Autoradiograph of labelled aminoacids

K_1 Alanine ;	K_3 Glutamic acid
K_2 Glycine ;	K_4 Aspartic acid



Samples were exposed under Ultra-violet light

Discussion

A perusal of the experimental results recorded in the foregoing tables 1 and 2, show that in every case the amino acids contents increased with time. The pH of the systems were also decreasing gradually and it was found that pH in the unphosphated system declined from 6·30 to 5·65 and in phosphated system the pH declined from 6·10 to 5·05. This declination in pH was due to the formation of amino acids as well as aliphatic acids From the activity measurement of carbon atom, in table III, shows that the total activity of the individual amino acids are distributed among -COOH and -$CHNH_2$- group. From the detail analysis it shows that the active carbon atom of glucose (-$C^{14}HO$) donated its activity among the first and second carbon atom in aminoacids molecule.

All the three reactions are taking place side by side as shown in the equations (1), (2) and (3) during the photo-chemical oxidation of glucose. Therefore during the evolution of normal carbon dioxide some labelled carbondioxide will also liberate from the system and for this reason the total activity is always less than the normal activity which was introduced to the system at the initial stage.

By comparing activity of - COOH and - $CHNH_2$- group we see that the activity of the first carbon atom (-$C^{14}OOH$) is more in comparison to the second carbon atom (-$C^{14}HNH_2$-). From the equations (10), (11) and (12) it seems that carbon monoxide is actually carrying out the activity of the labelled aldehydic carbon atom of glucose for the formation of labelled aminoacid. The cause of differences in activity for -$C^{14}HNH_2$- and -$C^{14}OOH$ group in aminoacid is still unknown.

TABLE I

Ultraviolet Lamp model No. S-4 Radiation range : 2900–3200 Å
Average temp : 35°C

$NiO + C_6H_{12}O_6$ + Glucose - 1 - C - 14

Hours exposed	Amino acids determined	Amount of amino acids mg/100 gm. sample					pH
		A	B	C	D	E	
30	A. B. C.	1·4	1·1	Trace	–	–	6·30
60	A. B. C. D.	1·7	1·3	1·2	0·9	Trace	6·20
90	A. B. C. D. E.	2·0	1·5	1·3	1·1	0·8	5·80
120	A. B. C. D. E.	2·3	1·8	1·6	1·3	0·1	5·50
150	A. B. C. D. E.	1·1	0·7	0·6	0·6	0·4	5·65

TABLE II

$NiO + C_6H_{12}O_6$ + Glucose - 1 - C - 14 + $CaHPO_4$

30	A. B. C.	1·7	1·3	0·6	–	–	6·10
60	A. B. C. D. E.	2·1	1·6	1·5	1·2	0·6	5·80
90	A. B. C. D. E.	2·4	1·8	1·6	1·5	1·0	5·45
120	A. B. C. D. E.	2·6	2·0	1·9	1·7	1·2	5·25
150	A. B. C. D. E.	1·3	0·9	0·8	0·9	0·6	5·05

Abbreviations :

A = Glycine, B = α – Alanine, C = Aspartic acid, D = Glutamic acid, E = Arginine.

TABLE III

Distribution of C^{14} in amino acids formed*

Carbon atom	Radioactivity	
	Phosphated	Unphosphated
Glutamic acid	c.p.m. × 10^5	c.p.m. × 10^5
Total	5·52	4·13
1 COOH	3·32	2·82
2 CH - NH_2	2·20	1·31
3 CH_2	Nil	Nil
4 CH_2	Nil	Nil
5 COOH	Nil	Nil
Glycine		
Total	6·32	4·80
1 COOH	3·64	2·90
2 CH – NH_2	2·68	1·90
Alanine		
Total	6·12	4·66
1 COOH	3·60	2·75
2 CH - NH_2	2·52	1·91
Aspartic acid		
Total	6·20	4·56
1 COOH	3·80	2·90
2 CH – NH_2	2·40	1·66

*The labelled amino acids obtained in this investigation by column-chromatographic technique were very small quantities for such detail studies. For the detail analysis some pure unlabelled amino were mixed (2 gm) with the respective amino acid fractions then proceeded for the detail analysis.

References

1. Baddiley. *J. Biol. Chem.*, **183**, 777, 1950.
2. Baly. "The Chemical Action of ultraviolet ray" *by* E. Carleton and A. Wells Alfred. *The Chemical Catalog Company, Inc. N. Y., U. S. A.*, 1925.
3. Banerjee and Dhar. *Ind. Chem. Soc. Ann. Convention*, 53, 1964.
4. ——— Under publication, *J. Ind. Chem. Soc.*, 1966.
5. Bernal. "The physical basis of life". *Routladge and Kegan Paul, London*, 1951.
6. Bernoulli and Cantieni. *Helv. Chem. Acta.*, **15**, 119, 1932.
7. Bernstein and Ballentine. *Rev. Sci. Instr.*, **21**, 158–162, 1950.
8. Briner, *et al.* *Helv. Chem. Acta.*, **23**, 826, 1940.
9. Compton and Langmuir *Rev. Modern. Phys.*, **2**, 124. 1930.
10. Dhar. "The chemical action of ultraviolet ray." *by* E. Carleton and A. Wells Alfred. *The Chemical Catalog Company, Inc. N. Y., U. S. A.*, 1925.
11. Dhar and Mukherjee. *J. Ind. Chem. Soc.*, **11**, 727, 1934.
12. Euler and Lindberg. *Biochem. Z.*, **39**, 410, 1912.
13. Fisher and Peters. *Brennstchem.*, **12**, 268, 1933.
14. Groth. *Z. Electrochem.*, **45**, 262, 1939.
15. Harrison, *et al.* *J. Chem. Phys.*, **30**, 355, 1959.
16. Herbst and Clarke. *J. Biol. Chem.*, **104**, 769, 1934.
17. Leob. *Ber.*, **46**, 490, 1913.
18. Miller. *J. Amer. Chem. Soc.*, **77**, 2351, 1955.
19. Mulder. *Ber. Chem. Ges.*, **9**, 1902, 1876.
20. Oparin. Origin of life. *Macmillan and Co., New York*, 1938.
21. Oro. *Nature*, **197**, 862, 1963.
22. Pavalovskaya, *et al.* International Symposium on the origin of life on Earth Mosco. *Pergamon Press. N. Y.*, 1957.
23. Price. *J. Chem. Phys.*, **4**, 147, 1937.
24. Sulaiman. *J. Ind. Chem. Soc.*, 1941.
25. Turner. "Farmer" Winter issue, **10**, 1949.
26. Van Slyke and Folch. *J. Biol. Chem.*, **136**, 509–541, 1940.
27. Von Erlenmeyer. *Ann. Chem.*, **191**, 576, 1878.
28. Wöhler, *et al.* "Hand book of chemistry" *by* L. Gmelin. *Printed for Cavendish Society (London)*. Vol. I, **378**, 380–81, 1852.
29. Zelikoff, *et al.* *J. Opt. Soc. Amer.*, **43**, 753, 1953.

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Separation and Determination of Constituents in some Ternary and Quaternary Mixtures by the Ring Oven Technique

By

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Abstract

Schemes have been worked out for the separation and subsequent determination of the constituents in the following mixtures of allied metals by the ring oven method :

(*i*) Copper, silver and gold

(*ii*) Zinc, cadmium and mercury

(*iii*) Silver, mercury, thallium and lead.

Individual metal ions are separated first in the form of thin rings. They are developed with suitable chromogenic reagents. The intensities of the developed rings are then compared visually with the standard rings, obtained with known amounts under identical conditions. Different organic solvents have been used for washing the spots.

Though the materials handled are very minute in quantity (a few drops only) the results are satisfactory and errors are within the permissible limits. A large number of artificially prepared mixtures of the above combinations of metals have been analysed by the proposed schemes. It has been found that the determinations are possible down to the following limits :

Au 3·00 μg ; Cu 0·95 μg ; Ag 0·81 μg ; Zn 0·98 μg ; Cd 1·69 μg ; Hg 3·02 μg ; Pb 3·11 μg ; Tl 1·53 μg.

The time required for each combination is not more than 70 minutes, excluding the time required for the preparation of the standard scales.

Introduction

Sometimes the separations and determinations of allied metal ions in mixtures are not possible with classical methods, specially when the amount of the samples available is small. In the present work schemes have been worked out for the separation and subsequent determination of the constituents in the following mixtures :

(*a*) Copper, silver and gold

(*b*) Zinc, cadmium and mercury

(*c*) Silver, mercury, thallium and lead.

The constituents in the above combinations are first separated in the form of thin rings in filter paper circles. These rings are developed with suitable chromogenic reagents and the intensities of each of the metal ions are compared visually with standard scales, similarly prepared with known amounts for its determination by the method of ring colorimetry[1].

Experimental

Apparatus : A Weisz ring oven (National Appliance, Co., U. S. A.) was used with a suitable power unit to yield 25 volts from 220 V/50 cycles a.c. mains. A self-filling capillary pipette, capacity ca. 1·5 µl was used throughout the procedure.

Filter Paper : Filter paper circles (diameter 55 mm) of Whatman No. 1 (chromatography grade) were used throughout and they have been termed as paper in the procedures.

Precipitant : 1% aqueous w/v sodium diethyldithiocarbamate.

Solvents for washing :

(*i*) Ethylacetate (B. D. H. AnalaR)

(*ii*) *n*-Butanol (B. D. H. AnalaR)

(*iii*) A 20% mixture of acetylacetone in chloroform (B. D. H. AnalaR)

(*iv*) A 20% aqueous pyridine solution (B. D. H. AnalaR)

(*v*) A 5 : 2 mixture of chloroform and acetone (B. D. H. AnalaR)

(*iv*) A 1 : 10 mixture of 3N acetic acid and 60% ethanol (B D. H. AnalaR).

Developing Reagents :

Benzidine : 0·05% w/v in 4M acetic acid

Cadion 2B : 0·1% w/v in ethanol

Dithizone : 0·05% w/v in carbon tetrachloride

Hydrogen sulphide : A saturated solution in acetone

4-(2-pyridylazo) resorcinol : 0·1% w/v aqueous solution

Rubeanic acid : 1% w/v in alcohol

Sodium rhodizonate : A saturated solution containing 1% acetic acid.

Metal solutions : Solutions of the following were prepared using reagent grade chemicals : nitrates of silver (I), lead (II), copper (II), mercury (II), cadmium (II), zinc (II), and chlorides of gold (III) and thallium (III). These solutions were standardised by the usual methods and diluted to 0·1M in each case. Suitable mixtures were prepared by mixing varying volumes of metal solutions. For spotting the solution on the filter paper, the self-filling pipette of volume ca. 1·5 µl was used. In preparing the standard scales the same self-filling pipette was used.

Procedure

(*a*) *Separation and Determination of Copper, Silver and Gold*

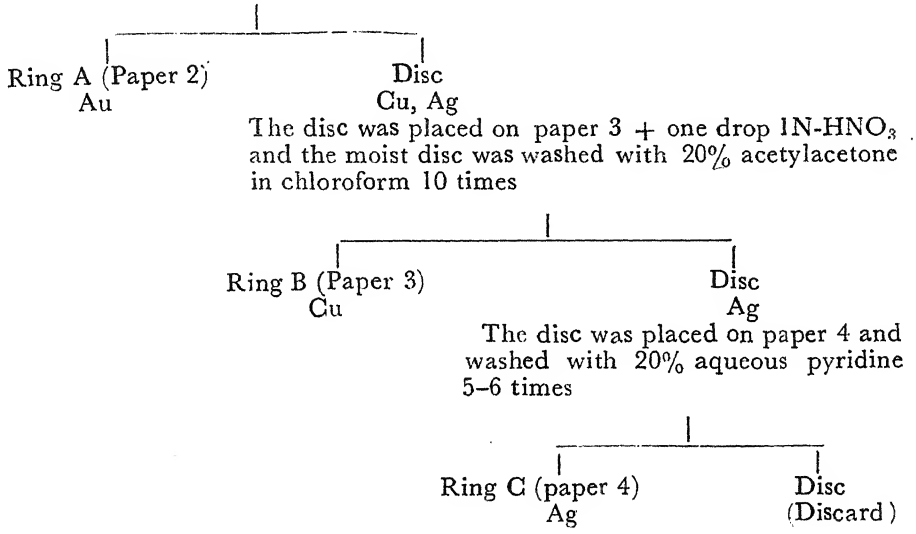
The mixture solution was spotted on the marked paper 1, dried on the hot ring oven and the spot was punched out. It was termed as disc. The disc was placed on paper 2 and it was moistened with one drop of 1N-HNO_3. The moist disc was washed 5 to 6 times with *n*-butanol. Gold (III) was transferred to ring zone (A). The disc was kept on paper 3, one drop 1N-HNO_3 was added to the disc. Then the moist disc was washed 10 times with 20% acetylacetone in chloroform to transfer copper (II) to ring zone (B). The disc containing only silver (I)

was kept on paper 4 and silver (I) was transported to ring zone (C) by washing the disc with 20% aqueous pyridine. The papers 2, 3 and 4 respectively containing gold, copper and silver were developed in the ring zones with reagents described in table 1. The separation is schematically described in Chart I.

CHART 1

Separation of Copper, Silver and Gold

The mixture was spotted on a marked paper 1, dried and punched out (disc). Disc was placed on the paper 2 and one drop 1N-HNO_3 added and the moist disc was washed with *n*-butanol (5-6 times)



(*b*) *Separation and Determination of Zinc, Cadmium and Mercury*

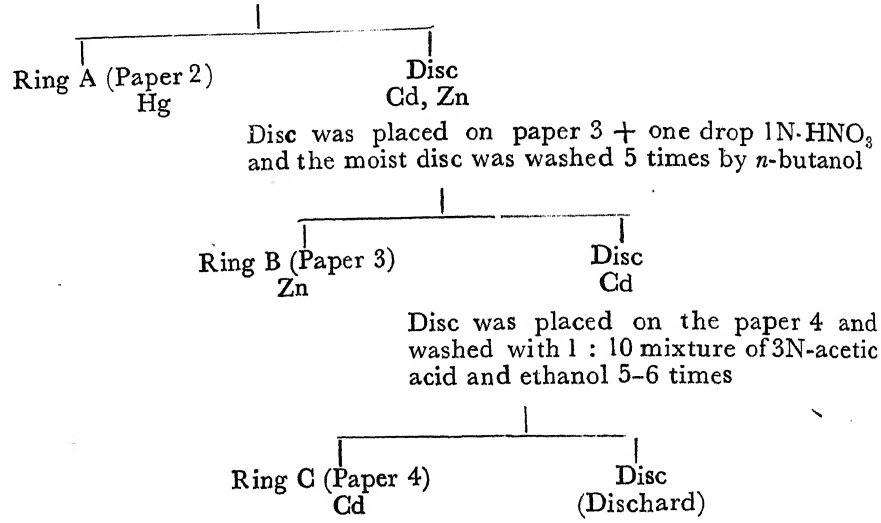
The mixture was spotted on a marked centre of paper 1, dried on the ring oven and punched out (disc). The disc was placed and another paper 2, one drop 1N-HNO_3 was added to the spot and the moist spot was washed with 20% acetylacetone in chloroform. Mercury (II) was transported to ring zone (A). The disc containing zinc and cadmium was kept on paper 3, again one drop 1N-HNO_3 was added to the disc and the moist disc was washed with *n*-butanol for 6 times to transfer zinc (II) to ring zone (B). The disc contained only cadmium, it was then kept on paper 4 and washed with a (1 : 10) mixture of 3N-acetic acid and 60% ethanol 8 times. Cadmium (II) migrated to ring zone (C). The disc was then discarded.

The papers 2, 3, and 4 containing respectively mercury, zinc and cadmium were developed in the ring zones with the reagents described in table 1. The separation has been described in Chart 2.

CHART 2

Separation of Zinc, Cadmium and Mercury

Mixture was spotted on paper 1, dried and punched out (disc). The disc was kept on paper 2 + one drop 1N-HNO_3 and the moist disc washed with 20% acetylacetone in chloroform (5-6 times)



(c) Separation and Determination of Silver, Mercury, Thallium and Lead

The mixture was spotted on the marked centre of paper 1, dri d over the ring oven aud punched out (disc). The disc was kept on another paper 2 on the ring oven, and was washed 6-8 times with ethylacetate. Mercury (II) was thus transported to ring zone (A). The disc containing the remaining metal ions kept on paper 3, one drop 1N-HNO_3 was added and the moist disc was washed 5-6 times with 20% acetylacetone in chloroform. Thus thallium (III) was washed to ring zone (B). The disc containing silver (I) and lead (II) was taken in the glass holder and treated with 3 drops of 1% aqueous sodium diethyldithiocarbamate and dried in the ring oven The disc was placed on paper 4 and washed 5 times with 5 : 2 mixture of chloroform and acetone. Thus lead (II) was transferred to ring zone (C). The disc now contained only silver (I). It was then placed on paper 5 and washed 3-4 times with 20% aqueous pyridine for complete washing to ring zone (D). The disc was then discarded.

The papers 2, 3, 4 and 5 respectively containing mercury, thallium, lead and silver, were developed in the ring zones with the reagents described in table 1. The separation is described below, in Chart 3.

CHART 3

Separation of Silver, Mercury, Thallium and Lead

Mixture was spotted on paper 1, dried and punched out (disc). Disc was placed on paper 2 and washed 6–8 times by ethylacetate.

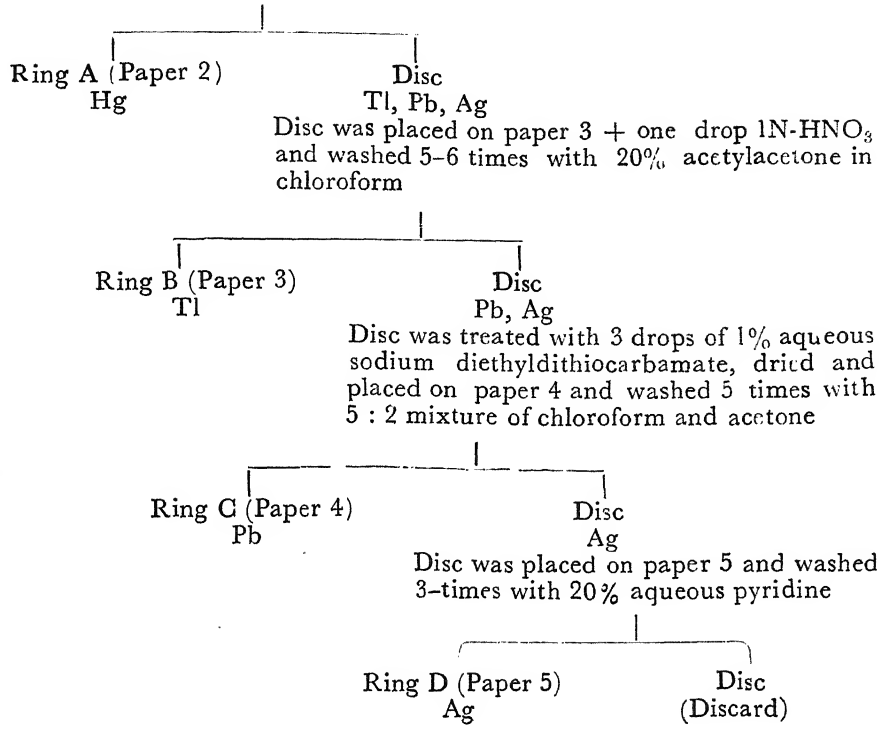


TABLE 1

Developing Reagents Employed

Metal ion	Reagent	Colour	Reference
Cadmium (II)	Cadion 2B	Pinkish-red	(2)
Copper (II)	Rubeanic acid	Olive green to black	(3)
Gold (III)	Benzidine	Blue	(3)
Lead (II)	Sodium rhodizonate	Blue - violet-red	(3)
Mercury (II)	Dithizone	Orange	(3)
Silver (I)	Hydrogen sulphide in acetone	Dark brown to black	(3)
Thallium (III)	4–(2-pyridylazo) resorcinol	Red	(4)
Zinc (II)	Dithizone	Purple-red	(3)

For the quantitative determinations, appropriate standard scales were prepared with each of the individual cations separately. The standard rings (5) were prepared with even numbers of drops of the standard solutions, by using the self-filling micropipette (1·5 µl), and were numbered II. IV, VI, VIII and X and contained 2, 4, 6, 8 and 10 drops of solution respectively. A proper care has to be taken that the spot may not be too large in diameter when several drops were added. This was achieved by drying the previous spot before applying a fresh drop. Similarly the test rings were prepared with 1, 3, 5, 7 and 9 drops and were numbered I, III, V, VII and IX respectively. The total number of test rings were compared with total number of standard rings.

The amounts of Cd, Cu, Au, Pb, Hg, Ag, Tl and Zn were determined by visual comparison of the intensities of the rings formed from test drops with those formed the standard drops (5, 6). The amount of each constituent in the rings is given by :

$$C_a = C_s \cdot n_s/n_a$$

where, C_s is the concentration of the standard solution, n_a the volume of sample solution and n_s the volume of standard solution used in matching ring.

If the intensities of the colour of the test rings do not fall within the standard scale then the concentration of the test solution may be altered by evaporation or dilution with water, as necessary.

A large number of artificial mixtures of each set were analysed. It has been found that determinations are possible down to the following limits when present in a drop of the solutions.

Ag 1·6 µg ; Au 3·00 µg ; Cd 1·69 µg ; Cu 0·95 µg ; Hg 3·04 µg ; Pb 3·11 µg ; Tl 1·53 µg ; Zn 0·98 µg.

Some of the typical results for the separation of mixtures (*a*), (*b*) and (*c*) are shown in table 2, (*a*), (*b*) and (*c*) respectively.

TABLE 2 (*a*)

Results of the Determinations (*a*)

Taken mg/ ml			Found mg/ml			% Error		
Au	Cu	Ag	Au	Cu	Ag	Au	Cu	Ag
1·97	0·3175	1·08	1·97	0·3177	1·078	0·00	+0·06	−0·18
2·75	0·4445	1·51	2·76	0·4448	1·510	+0·36	+0·06	0·00
3·54	0·5715	1·94	3 55	0·5719	1·942	+0·28	+0·07	+0·10

TABLE 2 (*b*)

Results of the Determinations (*b*)

Taken mg/ml			Found mg/ml			% Error		
Hg	Zn	Cd	Hg	Zn	Cd	Hg	Zn	Cd
1·40	0·98	1·12	1·40	0·92	1·12	0·0	−6·1	0·0
2·00	1·30	2·25	2·01	1·31	2·24	+0·5	+0·8	−0·4
3·00	1·96	3·37	2·81	1·83	3·15	−6·3	6·6	-6·3

TABLE 2 (*c*)

Results of the Determinations (*c*)

No.	Metal ion	Taken mg/ml	Found mg/ml	% Error
I	Hg	2·00	2·04	+2·0
	Tl	1·00	1·02	+2·0
	Pb	2·50	2·64	+5·6
	Ag	0·60	0·64	+6·6
II	Hg	3·00	3·20	+6·6
	Tl	2·00	2·04	+2·0
	Pb	4·50	4·30	-4·4
	Ag	0·80	0·76	-5·0
III	Hg	4·00	4·02	+0·5
	Tl	3·00	2·86	-4·6
	Pb	5·50	5·29	-3·8
	Ag	1·00	0·97	-3·0

Results of the determinations are satisfactory, and the error lies within the permissible limits. The time required for separation and determination of the constituents of the above combinations was about 60–70 minutes, excluding the time required for the preparation of the standard scales.

References

1. Stephen, W. I. *Ind. Chemist*, **36** : 408, 1960.
2. Weisz, H. Microanalysis by the Ring Oven Technique, Pergamon, Oxford, 1961.
3. Feigl, F. Spot Tests in Inorganic Analysis, 5th ed., Elsevier, Amsterdam, 1958.
4. Dwivedi, C. D., Munshi, K. N. and Dey, A. K. *J. Inorg. Nucl. Chem*, **28**, 245, 1966.
5. Knödel, W. and Weisz, H. *Mikrochim. Acta*, 417, 1957.
6. Weisz, H. *Mikrochim. Acta*, 460, 785, 1954.

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Studies on the Ilkovic Equation with a Dropping Mercury Electrode at 45°

By

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[Received on 30th August, 1967]

Abstract

Polarograms of Tl (I) over a wide range of concentrations (0·5 mM to 40 mM) have been recorded at a dropping mercury electrode bent at 45° in 2 M KNO_3 under varying heights of mercury column. The applicability of a few empirical equations suggested by several workers has been critically examined. Our results show that the original Ilkovic equation is well obeyed with almost constancy in $i_d/h^{\frac{1}{2}}$. The process is diffusion controlled. The half-wave potential ($E_{\frac{1}{2}}$) is shifted to more negative values with increase in the concentration of the depolarizer. An irreversible behaviour is depicted at higher cencentrations of the electroreducible ions

Introduction

D. Ilkovic[1] from theoretical considerations established the relation

$$i = A\,C\,m^{2/3}\,t^{1/6} \qquad (i)$$

The equation was modified[2-4] in the form

$$i = A\,C\,m^{2/3}\,t^{1/6} + B\,C\,m^{1/3}\,t^{1/3} \qquad (ii)$$

$$i = a\,t^{1/6} + b\,t^{1/3} \qquad (iii)$$

where A and B are constants depending on the dielectric constant of the medium and

$$a = A\,C\,m^{2/3}$$

$$b = B\,C\,m^{1/3}$$

while the other notations have their usual significance.

Several workers[5,6] have established the validity of the Ilkovic equation using vertical dropping mercury electrode. In our previous investigation[7], it has been shown that original Ilkovic equation is valid with a horizontal dropping mercury electrode. But the work of Smoler[8] who used capillaries horizontally and at 45°, no reference seems to be available in literature dealing with the study of C-V curves with an inclined dropping mercury electrode (d.m.e.). Therefore, the present study has been taken up with a view to seeing the validity of Ilkovic equation and critically examining the other empirical equations put forward by the previous workers. The studies have been made for Tl (I) over a wide range of concentrations in 2 M KNO_3 as the base electrolyte at five different heights of mercury column.

Experimental

Solution of $TlNO_3$ (E. Merck) was prepared in conductivity water. KNO_3 was a BDH product. Triton X-100 (0·005%) was used as maximum suppressor in the solutions subjected to the study of C-V curves. Pure distilled mercury was used for the d.m.e.

The design of the cell and d.m.e. at 45° is depicted in fig. 1. A wooden support was given to the capillary so as to minimise its to and fro movement. Lange's polarometer in conjunction with Multiflex galvanometer MGF_2 was used for the record of voltage and current respectively. Deoxygenation of the solutions was done by purified hydrogen. Saturated NH_4NO_3 salt bridge was used for connecting the saturated calomel electrode (S.C.E.) with the other half cell. The corrected heights (h_{corr}) were obtained by subtracting h_{back} = 1·50 cms from the original heights. The temperature during these measurements was maintained at 30°C.

Results

TABLE I

Effects of Mercury Pressure on m and t

The data were secured at $E_{d.e.}$ = − 0·5 V *vs.* S.C.E. in 2 M KNO_3 at 30°C.

S. No.	h_{corr} cms.	$h^{\frac{1}{2}}$ cms.	t sec.	m mg/sec.	mt, mg	100 m / h_{corr}	$h_{corr}\ t$
a	47	6·85	2·68	1·10	2·94	2·34	125·9
b	42	6·48	2·92	0·98	2·86	2·33	122·6
c	37	6·08	3·28	0·88	2·88	2·37	121·3
d	32	5·65	3·74	0·75	2·80	2·35	119·7
e	27	5·19	4·48	0·65	2·91	2·40	120·9

TABLE II

Conc. of Tl (I) = 0·5 mM.

Heights	$E_{\frac{1}{2}}$ *vs.* S.C.E. volts	i_d μ amp	Slope V	$i_d/h^{\frac{1}{2}}$
a	−0·465	2·0	0·055	0·297
b	−0·465	1·9	0·055	0·293
c	−0·470	1·8	0·050	0·294
d	−0·470	1·7	0·050	0·301
e	−0·470	1·6	0·050	0·307
	Conc. of Tl (I) = 1 mM			
a	−0·475	4·0	0·055	0·59
b	−0·470	3·7	0·055	0·57
c	−0·470	3·4	0·056	0·56
d	−0·470	3·2	0·055	0·57
e	−0·475	2·9	0·057	0·57
	Conc. of Tl (I) = 5 mM			
a	−0·480	19·3	0·08	2·82
b	−0·485	18·5	0·08	2·84
c	−0·490	17·4	0·075	2·86
d	−0·490	16·2	0·08	2·85
e	−0·490	14·8	0·08	2·84

Heights	$E_{\frac{1}{2}}$ *vs.* S.C.E. volts	i_d μ amp	Slope V	$i_d/h^{\frac{1}{2}}$
		Conc. of Tl (I) = 10 mM		
a	-0·490	39·8	0·08	5·80
b	-0·490	37·6	0·08	5·80
c	-0·490	35·2	0·08	5·78
d	-0·490	32·7	0·084	5·77
e	-0·490	29·8	0·08	5·76
		Conc. of Tl (I) = 20 mM		
a	-0·51	76·0	0·085	11·13
b	-0·51	71·2	0·090	11·07
c	-0·51	68·2	0·090	11·21
d	-0·50	62·9	0·092	11·11
e	-0·49	50·8	0·088	9·87
		Conc. of Tl (I) = 40 mM		
a	-0·52	155·2	0·11	22·60
b	-0·52	144·6	0·14	22·34
c	-0·52	135·7	0·11	22·30
d	-0·52	125·1	0·12	22·07
e	-0·52	116·3	0·13	22·30

Discussion

Perusal of the data in table II reveals that the half-wave potential ($E_{\frac{1}{2}}$) of Tl (I) is shifted to more negative values with increasing concentrations of the depolarizer. Tl does not undergo complexation in KNO_3. Obviously, the shift of $E_{\frac{1}{2}}$ towards more negative potentials is due to some additional factors other than complexation. Even after application of iR drop correction, the $E_{\frac{1}{2}}$ of the depolarizer shifts to more negative values at higher concentrations. This is owing to the irreversible behaviour of the depolarizer at higher concentrations.

However, at lower concentrations, the raction appears to be reversible but with the increasing concentration of the electroactive ions, the reversible character diminishes as shown by the slope values of plots from $E_{d.e.}$ *vs.* log i/i_d-i. An appreciable divergence between the experimental and theoretical slopes is found which implies that the reaction is irreversible[9]. The study of $E_{\frac{1}{2}}$ further supplements the irreversible behaviour at higher concentrations. Any appreciable variation[10] of $E_{\frac{1}{2}}$ with concentration can be taken as a conclusive proof of the irreversibility of the half reaction. The plot of log t *vs.* $E_{\frac{1}{2}}$ which happens to be a straight line further substantiates the previous conclusions.

In addition to the major contributions towards changes in $E_{\frac{1}{2}}$ because of irreversibility, other minor factors *viz.* change of activity coefficient, ionic strength, ionic environment and variation of fluid potential may be operative though to the extent of a few millivolts[11] only.

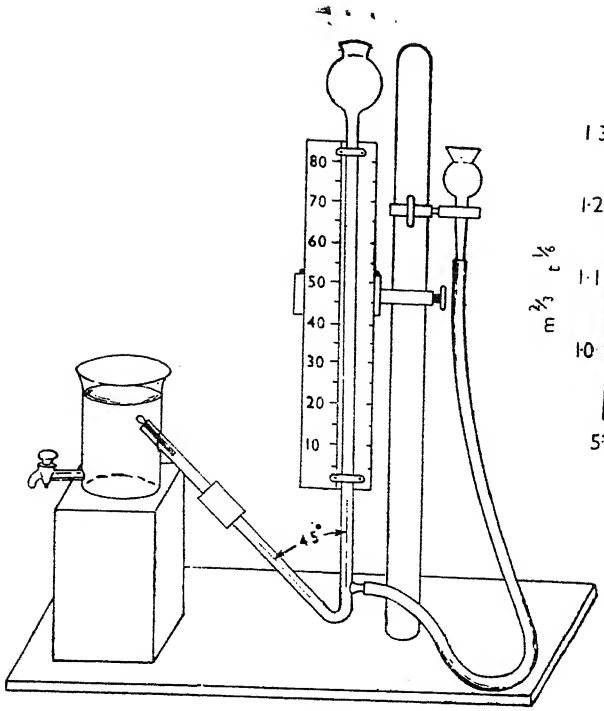

Fig. 1

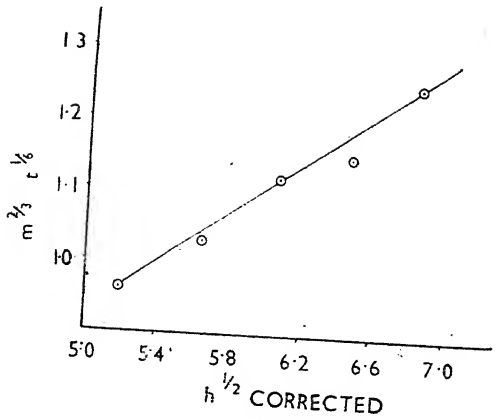

Fig. 2

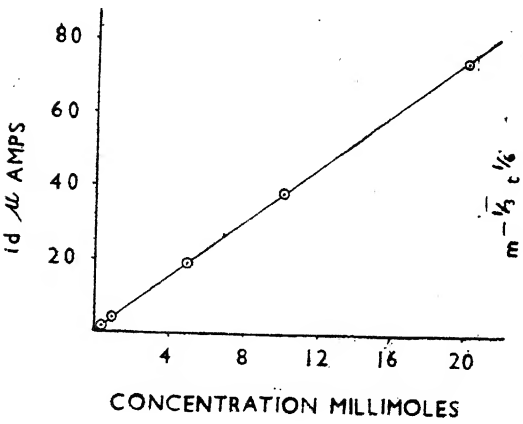

Fig. 4

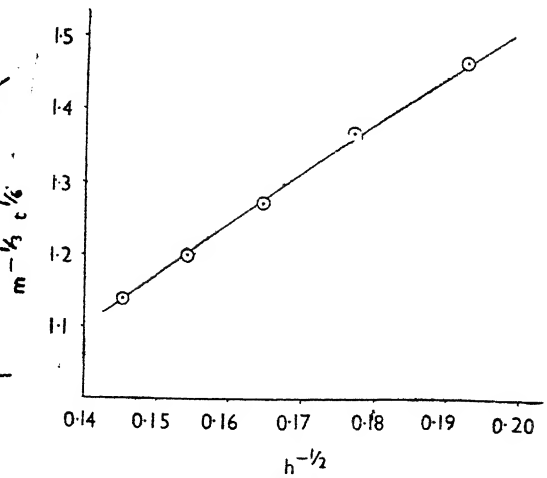

Fig. 3

Fig. 1. Diagrammatic sketch of the cell and d.m.e. at 45°

Fig. 2. Graph between $h^{\frac{1}{2}}$ and $m^{\frac{2}{3}}\ t^{\frac{1}{6}}$

Fig. 3. Plot depicting $h^{-\frac{1}{2}}$ *vs.* $t^{\frac{1}{6}}\ m^{-\frac{1}{3}}$

Fig. 4. Graph showing the linearity of i_d with C at 47 cms. height of mercury column.

The variation of m and t with mercury pressure is described in table I. The values of mt, $100\ m/h_{corr}$ and ht are almost constant within experimental errors.

In order to test the validity of the various equations put forward by several workers, the following approach was adopted. A plot $i\ t^{-1/6}$ against $t^{1/6}$ was not found to be linear. Our results are in conformity with those of Meites *et al.*[12] and Bresle[13]. The present results do not support the inclusion of a correction term $b\ t^{1/3}$ (cf. equation *iii*) in the original Ilkovic equation. Similarly on plotting $i\ t^{-1/6}$ against $t^{-\frac{1}{2}}$ and $i\ t^{1/3}$ against $t^{\frac{1}{2}}$ no linearity has been found between these parameters. Hence, it is inferred that the equation due to Newcombe and Woods[14] in the form

$$i = \alpha\ t^{\frac{1}{2}} + \beta\ t^{-1/3}$$

is also not applicable to the data obtained with d.m.e. bent at 45°. However, the plots of $h^{\frac{1}{2}}_{corr}$ *vs.* $m^{2/3}\ t^{1/6}$ and $h^{-\frac{1}{2}}_{corr}$ *vs.* $m^{-1/3}\ t^{1/6}$ are linear (figs. 2, 3). Similarly a linear plot is obtained between i_d and C (fig. 4). In view of the above observations we find that the process is diffusion controlled. According to the Ilkovic equation

$$\frac{i_d}{h^{\frac{1}{2}}_{corr}} = k_1$$

where k_1 is constant if nothing in varied except h_{corr}. The plot of i_d *vs.* $h^{\frac{1}{2}}_{corr}$ is also linear. The constancy of $i_d/h^{\frac{1}{2}}_{corr}$ is apparent from table II which shows explicitly that the original Ilkovic equation is best applicable also for a d.m.e. inclined at 45°.

Acknowledgements

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References

1. Ilkovic, D. *Coll. Czech. Chem. Comm.* **6**, 498, 1934.
2. Kambara, T., Suzuki M. and Tachi, I. *Bull. Chem. Soc., Japan*, **23**, 219, 1951.
3. Lingane, J. J. and Loveridge, B. A. *J. Am. Chem. Soc.*, **72**, 438, 1950.
4. Strehlow, H. and Stackleberg, M. von. *Z. Electrochem.*, **54**, 51, 1950.
5. Lingane, J. J. and Kolthoff, I. M. *J. Am. Chem. Soc.*, **61**, 825, 1939.
6. Lingane, J. J. and Loveridge, B. A. *J. Am. Chem. Soc.*, **66**, 1425, 1944.
7. Lal, S., Jha, S. K. and Srivastava, S. N. Unpublished results.
8. Smoler, I. *Chem. Listy*, **47**, 1667, 1953.
9. Meites, L. Polarographic Techniques. *Interscience Pub. N. Y.* 264, *2nd Ed.* 1965.
10. *ibid.* 228.
11. Deford, D. D. and Donald, L. Anderson, *J. Am. Chem. Soc.*, **72**, 3918, 1950.
12. Meites, L. and Meites, T. *J. Am. Chem. Soc.*, **72**, 4843, 1950.
13. Bresle, A. *Acta Chem. Scand.*, **10**, 935, 1956.
14. Newcombe, R. J. and Woods, R. Advances in Polarography. Vol. I *Pergamon Press*, 375, 1960.

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On Generalized Double Hypergeometric Function

By

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[Received on 24th August, 1967]

Abstract

The object of this paper is to obtain certain recurrence relations for the generalised double hypergeometric function, which are the generalisation of the recurrence relations for hypergeometric function recently obtained by the author.

1. The object of this paper is to obtain certain recurrence relations for the generalized double hypergeometric function

$$F^{(5)}\left[\begin{matrix} a, b : d \\ c : e \end{matrix} ; x, y\right] = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(a)_{r+s}\,(b)_{r+s}\,(d)_r}{(c)_{r+s}\,(e)_r\,\lfloor r \rfloor \lfloor s}\, z^r y^s$$

which is a special case of the Kampe de Feriet's hypergeometic function of two variables and in Kampe de Feriet's notation, it is the function

$$F\begin{bmatrix} 2 & c\,,\ d & \\ 1 & a\,,\ b & x \\ 1 & e & y \\ 1 & a',\ b' & \end{bmatrix}$$

These relations have been obtained in section A, with the help of two lemmas about a transform given by Banerjee and defined by the integral equation

$$\phi[f(t)\,;\, m, s, k, \mu] = \int_0^{\infty} (2pt)^{m-\mu-1}\, e^{-pt\left(\frac{s}{2}-1\right)}\, W_{k,\mu}(2pt)\, f(t)\, dt \qquad (1{\cdot}1)$$

In section B, these lemmas have been illustrated with an example, gives us the relations for the generalized double hypergeometric function, which are the generalisation of the results for the hypergeometric function ${}_2F_1$ obtained recently by the author[2]

We shall write $f(t) \in c(\sigma, \rho\,;\, \alpha)$, if

$$f(t) = \begin{cases} O(t^{\sigma}) & \text{for small } t. \\ O(e^{\alpha t}\, t^{\rho}) & \text{for large } t. \end{cases}$$

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Section A

2. *Lemma* I

If $f(t) \in c(\sigma, \rho, \alpha)$, $Re(\sigma + m - \mu \pm \mu + \frac{1}{2}) > 0$ and $Re(p) > Re(\alpha)$ then

$$\sum_{r=0}^{n} (-1)^{n-r}\, {}^{n}C_{r}\, \frac{\Gamma(\mu-k+n-r+\frac{1}{2})}{\Gamma(\mu-k+\frac{1}{2})}\, \phi\left[f(t)\,;\, m,\, s,\, k-\frac{n}{2}+r,\, \mu+\frac{n}{2}\right]$$

$$= \phi[f(t)\,;\, m,\, s,\, k,\, \mu] \qquad (2\cdot1)$$

Proof:

Recently, author has proved in[2], that

$$\sum_{r=0}^{n} (-1)^{n-r}\, {}^{n}C_{r}\, \frac{\Gamma(\mu-k-r+\frac{1}{2})}{\Gamma(\mu-k-n+\frac{1}{2})}\, W_{k+r,\mu}(2\,at) = (2\,at)^{n/2} W_{k+\frac{n}{2},\,\mu-\frac{n}{2}}(2\,at) \qquad (2\cdot2)$$

Now multiplying both the sides of (2·2) by $(2\,at)^{m-\mu-1}\, e^{-\frac{1}{2}at}\, (s/2-1)\, f(t)$, integrating between the limits zero and infinity, replacing a, k and μ by p, $k-n/2$ and $\mu+n/2$ respectively and interpreting the result with the help of (1·1), we get the required result (2·1)

Lemma II

If $f(t) \in c(\sigma, \rho\,;\, \alpha)$, $Re[\sigma + m - \mu \pm \mu + \frac{1}{2}] > 0$ and $Re(p) > Re(\alpha)$ then

$$\sum_{r=0}^{n} (-1)^{n-r}\, {}^{n}C_{r}\, \phi\left[f(t)\,;\, m,\, s,\, k+\frac{r}{2}+\frac{n}{2},\, \mu+\frac{r}{2}-\frac{n}{2}\right]$$

$$= \frac{\Gamma(\mu-k+\frac{1}{2})}{\Gamma(\mu-n-k+\frac{1}{2})}\, \phi\,[f(t)\,;\, m,\, s,\, k,\, \mu) \qquad (2\cdot3)$$

Proof:

Proof of this lemma is on the same lines as that of lemma I, except that instead of (2·2), we use the result[2]

$$\sum_{r=0}^{n} (-1)^{n-r}\, {}^{n}C_{r}\, (2\,at)^{n/2-r/2}\, W_{k+r/2,\,\mu+r/2}(2\,at)$$

$$= \frac{\Gamma(\mu+n-k+\frac{1}{2})}{\Gamma(\mu-k+\frac{1}{2})}\, W_{k-n/2,\,\mu+n/2}(2\,at) \qquad (2\cdot4)$$

3. *Relations for the generalized hypergeometric function*

If we take $f(t) = M_{\lambda,v}(2\,\alpha\,t)$, so that [3, p. 437]

$$\phi[M_{\lambda,v}(2\,\alpha\,t)\,;\, m,\, s,\, k,\, \mu] = \frac{\Gamma(v \pm \mu - \mu + m + 1)}{\Gamma(3/2 - k + m + v - \mu)}$$

$$\frac{1}{2}\left(\frac{\alpha}{p}\right)^{v+\frac{1}{2}}\, F^{(5)}\left[\begin{matrix} 1+v+m-\mu \pm \mu : \frac{1}{2}+\lambda+v \\ \frac{s}{2}-k+m+v-\mu : 2\,v+1 \end{matrix}\;;\; -\frac{\alpha}{p},\, 1-\frac{s}{4}+\frac{\alpha}{2p}\right]$$

Using this value of $f(t)$ in the results (2·1) and (2·3), we get the following relations after making suitable adjustment in parameters.

$$\sum_{r=0}^{n} (-1)^{n-r}\, {}^{n}C_{r}\, \frac{\Gamma(c-b+n-r)\,\Gamma(c)\,\Gamma(b-n)}{\Gamma(c-b)\,\Gamma(c-r)\,\Gamma(b)}$$

$$F^{(5)}\left[\begin{matrix} a,\ b-n : d \\ c-r : e \end{matrix}; x, y\right] = F^{(5)}\left[\begin{matrix} a,\ b : d \\ c : e \end{matrix}; x, y\right] \tag{3·1}$$

and

$$\sum_{r=0}^{n} (-1)^{n-r}\, {}^{n}C_{r}\, \frac{\Gamma(b+n-r)\,\Gamma(c)\,\Gamma(c-b-n)}{\Gamma(c-r)\,\Gamma(b)\,\Gamma(c-b)}$$

$$F^{(5)}\left[\begin{matrix} a,\ b+n-r : d \\ c-r\ ;\ e \end{matrix}; x, y\right] = F^{(5)}\left[\begin{matrix} a,\ b : d \\ c : e \end{matrix}; x, y\right] \tag{3·2}$$

4. *Particulars Cases*

(*i*) On taklng $d = e$ in the above results and using on elementary expansion

$${}_2F_1[a, b\ ;\ c\ ;\ x+y] = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(a)_{r+s}\,(b)_{r+s}\,(d)_{r}}{(c)_{r+s}\,(e)_{r}\,\lfloor r\ \rfloor s}\, x^{r} y^{s}$$

we get, the following relations for hypergeometric function given by author[2]

$$\sum_{r=0}^{n} (-1)^{n-r}\, {}^{n}C_{r}\, \frac{\Gamma(c-b+n-r)\,\Gamma(c)\,\Gamma(b-n)}{\Gamma(c-b)\,\Gamma(c-r)\,\Gamma(b)}\, {}_2F_1\left(\begin{matrix} a,\ b-n \\ c-r \end{matrix}; z\right)$$

$$= {}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix}; z\right) \tag{4·1}$$

and

$$\sum_{r=0}^{n} (-1)^{n-r}\, {}^{n}C_{r}\, \frac{\Gamma(b+n-r)\,\Gamma(c)\,\Gamma(c-b-n)}{\Gamma(c-r)\,\Gamma(b)\,\Gamma(c-b)}\, {}_2F_1\left(\begin{matrix} a,\ b+n-r \\ c-r \end{matrix}; z\right)$$

$$= {}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix}; z\right) \tag{4·2}$$

(*ii*) On taking $n = 1$, results (3·1) and (3·2) gives us the following recurrence relation for the generalized double hypergeometric function

$$(b-c)\, F^{(5)}\left[\begin{matrix} a,\ b : d \\ c+1 : e \end{matrix}; x, y\right] + (c)\, F^{(5)}\left[\begin{matrix} a,\ b : d \\ c : e \end{matrix}; x, y\right]$$

$$= (b)\, F^{(5)}\left[\begin{matrix} a,\ b+1 : d \\ c+1 : e \end{matrix}; x, y\right]$$

(*iii*) On taking $a = c$ and using the result [1, p. 224] results (3·1) and (3·2) gives us

$$\sum_{r=0}^{n} (-1)^{n-r}\, {}_nC_r \frac{\Gamma(a-b+n-r)\,\Gamma(a)\,\Gamma(b-n)}{\Gamma(a-b)\,\Gamma(a-r)\,\Gamma(b)}\, F^{(5)}\left[\begin{matrix} a, & b-n : d \\ & a-r : e \end{matrix}; x, y\right]$$

$$= F_2[b, d, b, e, b, x, y] \tag{4·4}$$

and

$$\sum_{r=0}^{n} (-1)^{n-r}\, {}_nC_r \frac{\Gamma(b+n-r)\,\Gamma(a)\,\Gamma(a-b-n)}{\Gamma(a-r)\,\Gamma(b)\,\Gamma(a-b)}\, F^{(5)}\left[\begin{matrix} a, & b+n-r; \\ & a-r \quad : \end{matrix} x, y\right]$$

$$= F_2[b, d, b, e, b\,;\, x, y] \tag{4·5}$$

(*iv*) On taking $n = 1$, relations (4·4) and (4·5) gives us

$$(b-1)\, F_2[b, d, b, e, b\,;\, x, y] + (a-b).\, F_2[b-1, d, b-1, e, b-1\,;\, x, y]$$

$$= (a-1)\, F^{(5)}\left[\begin{matrix} a, & b-1 : d \\ & a-1 : e \end{matrix}; x, y\right] \tag{4·6}$$

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References

1. Erdelyi, A. Higher transcedental functions. **Vol. I**, *McGraw Hill, New York*, 1954.
2. Pathan, M. A. Certain recurrence relations. *Communicated for publication.*
3. Srinivasan. On the generalised Meijer Transform. *Bull. de L' Academic Polonaise des sciences*, **11**(7), 431-440, 1963.

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On the solution of dual integral equations

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1. Introduction : We define the two functions $G(x)$ and $F(x)$ in terms of a third function $f(y)$ by the equations

$$\left.\begin{aligned}\int_0^\infty y^\alpha f(y)\, J_\nu(xy)\, dy &= G(x)\\ \int_0^\infty f(y)\, J_\nu(xy)\, dy &= F(x)\end{aligned}\right\} \tag{1·1}$$

A standard problem in the theory of dual integral equations is to determine the function $f(y)$ such that

$$G(x) = g(x)\ (0 \leqslant x \leqslant 1),\ F(x) = \varphi(x)\ (x > 1) \tag{1·2}$$

when the functions $g(x)$ and $\varphi(x)$ are prescribed.

The case in which $\varphi(x) = 0$, has been considered by several authors[1,2,3]. Recently Copson[4] has given a simple and elegent solution of this pair by a method which is a generalisation of an elementary method suggested by Sneddon[5].

The solution of the pair of equations (1·2) with $\alpha = \pm 1$ was given by Tranter[6] but his method is cumbersome. The pair with general values of α in the range $-2 < \alpha < 2$ was considered by Noble[7], who reduced the problem to that of solving an integral equation. Noble's analysis invoives considerable manipulation and cannot be regarded as elementary. More recently Williams[8] has derived the solution on the pair (1·2) by a formal application of Mellin's transform. The manipulation here is formally more simple because much of it can be absorbed in the calculation of Mellin transform.

Recently M. Lowengrub and I. N. Sneddon[9] have given the solution of the pair (1·1) for

$$G(x) = 0\ (0 \leqslant x \leqslant 1),\ F(x) = \varphi(x)\ (x > 1)$$

In the present paper we shall generalise the dual integral equations (1·1) and consider the equations

$$G(x) = g(x),\ (0 \leqslant x \leqslant 1),\ F(x) = 0,\ (x > 1). \tag{1·3}$$

2. Watson[10] has given a kernel $\varpi_{\mu,\nu}(x)$ and it has been shown by Bhatnagar[11] that this plays the role of a transform and is a generalisation of Hankel transform. Recently R. Narain[12] has shown that

(2·1) $$\widetilde{\omega}_{\mu,\nu}(x) \equiv \tfrac{1}{2}\sqrt{x}\, G^{2,0}_{0,4}\left(\frac{x^2}{16}\middle|\frac{\mu}{2}, \frac{\nu}{2}, -\frac{\mu}{2}, -\frac{\nu}{2}\right).$$

We define a pair of simultaneous equations

(2·2) $$\left.\begin{aligned}\int_0^\infty y^\alpha f(y)\, \varpi_{\mu,\nu}(xy)\, dy &= g(x), \quad 0 \leqslant x \leqslant 1\\ \int_0^\infty f(y)\, \varpi_{\mu,\nu}(xy)\, dy &= 0, \qquad\qquad x > 1\end{aligned}\right\}$$

in which the function $g(x)$ is a known function and the function $f(y)$ has to be determined.

The pair of equations (2·2) may be solved as follows :

The Mellin transform of the function $y^\alpha\, \varpi_{\mu,\nu}(xy)$ is

$$\widetilde{\omega}_\alpha(s) = \int_0^\infty y^{\alpha+s-1}\, \varpi_{\mu,\nu}(xy)\, dy$$

$$= \frac{4^{\alpha+s-\frac{1}{2}}\, \Gamma\left(\frac{\mu}{2}+\frac{\alpha}{2}+\frac{1}{4}+\frac{s}{2}\right)\Gamma\left(\frac{\nu}{2}+\frac{\alpha}{2}+\frac{1}{4}+\frac{s}{2}\right)}{x^{\alpha+s}\, \Gamma\left(\frac{\mu}{2}-\frac{\alpha}{2}+\frac{3}{4}-\frac{s}{2}\right)\Gamma\left(\frac{\nu}{2}-\frac{\alpha}{2}+\frac{3}{4}-\frac{s}{2}\right)},$$

as can be readily obtained by using the identity (2·1) and applying[13]. Putting $\alpha = 0$ in this expression, we see that the Mellin transform of $\varpi_{\mu,\nu}(xy)$ in the function

$$\widetilde{\omega}_0(s) = \frac{4^{s-\frac{1}{2}}\, \Gamma\left(\frac{\mu}{2}+\frac{1}{4}+\frac{s}{2}\right)\Gamma\left(\frac{\nu}{2}+\frac{1}{4}+\frac{s}{2}\right)}{x^s\, \Gamma\left(\frac{\mu}{2}+\frac{3}{4}-\frac{s}{2}\right)\Gamma\left(\frac{\nu}{2}+\frac{3}{4}-\frac{s}{2}\right)}$$

Now by the Faltung theorem for Mellin transforms[14], we have

$$\int_0^\infty y^\alpha f(y)\, \varpi_{\mu,\nu}(xy)\, dy = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)\, \widetilde{\omega}_\alpha(1-s)\, ds$$

$$\int_0^\infty (f(y)\, \varpi_{\mu,\nu}(xy)\, dy) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)\, \widetilde{\omega}_0(1-s)\, ds$$

where $F(s)$ is the Mellin transform of the function $f(y)$.

The dual integral equations (2·2) are therefore equivalent to the pair of equations.

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)\, \frac{\Gamma\left(\frac{\mu}{2}+\frac{\alpha}{2}+\frac{3}{4}-\frac{s}{2}\right)\Gamma\left(\frac{\nu}{2}+\frac{\alpha}{2}+\frac{3}{4}-\frac{s}{2}\right)4^{\alpha-s+\frac{1}{2}}}{\Gamma\left(\frac{\mu}{2}-\frac{\alpha}{2}+\frac{1}{4}+\frac{s}{2}\right)\Gamma\left(\frac{\nu}{2}-\frac{\alpha}{2}+\frac{1}{4}+\frac{s}{2}\right)x^{\alpha-s+1}}\, ds = g(x),$$

$$(0 \leqslant x \leqslant 1)$$

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)\ \frac{\Gamma\left(\frac{\mu}{2}+\frac{3}{4}-\frac{s}{2}\right)\Gamma\left(\frac{\nu}{2}+\frac{3}{4}-\frac{s}{2}\right)4^{-s+\frac{1}{2}}}{\Gamma\left(\frac{\mu}{2}+\frac{1}{4}+\frac{s}{2}\right)\Gamma\left(\frac{\nu}{2}+\frac{1}{4}+\frac{s}{2}\right)x^{-s+1}}\,ds = 0,\ (x>0)$$

If now we put

$$F(s) = 4^{s-\alpha-\frac{1}{2}}\ \frac{\Gamma\left(\frac{\mu}{2}+\frac{1}{4}+\frac{s}{2}\right)\Gamma\left(\frac{\nu}{2}+\frac{1}{4}+\frac{s}{2}\right)}{\Gamma\left(\frac{\mu}{2}+\frac{\alpha}{2}+\frac{3}{4}-\frac{s}{2}\right)\Gamma\left(\frac{\nu}{2}+\frac{\alpha}{2}+\frac{3}{4}-\frac{s}{2}\right)}\chi(s),$$

these equations reduce to

$$(3\cdot1)\quad \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma\left(\frac{\mu}{2}+\frac{1}{4}+\frac{s}{2}\right)\Gamma\left(\frac{\nu}{2}+\frac{1}{4}+\frac{s}{2}\right)x^{s-\alpha-1}}{\Gamma\left(\frac{\mu}{2}-\frac{\alpha}{2}+\frac{1}{4}+\frac{s}{2}\right)\Gamma\left(\frac{\nu}{2}-\frac{\alpha}{2}+\frac{1}{4}+\frac{s}{2}\right)}\chi(s)\,ds = g(x),\ (0\leqslant x\leqslant 1)$$

$$(3\cdot2)\quad \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma\left(\frac{\mu}{2}+\frac{3}{4}-\frac{s}{2}\right)\Gamma\left(\frac{\nu}{2}+\frac{3}{4}-\frac{s}{2}\right)x^{s-1}}{\Gamma\left(\frac{\mu}{2}+\frac{\alpha}{2}+\frac{3}{4}-\frac{s}{2}\right)\Gamma\left(\frac{\nu}{2}+\frac{\alpha}{2}+\frac{3}{4}-\frac{s}{2}\right)}\chi(s)\,ds = 0,\ (x>1)$$

Multiplying (3·1) by $x^{\alpha-\omega}$ where $R(s)-R(w)>0$, and integrating with respect to x from 0 to 1, we obtain the relation

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma\left(\frac{\mu}{2}+\frac{1}{4}+\frac{s}{2}\right)\Gamma\left(\frac{\nu}{2}+\frac{1}{4}+\frac{s}{2}\right)\chi(s)}{\Gamma\left(\frac{\mu}{2}-\frac{\alpha}{2}+\frac{1}{4}+\frac{s}{2}\right)\Gamma\left(\frac{\nu}{2}-\frac{\alpha}{2}+\frac{1}{4}+\frac{s}{2}\right)(s-\omega)}\,ds = G(\alpha-\omega+1),$$

where

$$G(s) = \int_0^1 g(x)\,x^{s-1}\,dx$$

is the Mellin transform of the function

$$y(x) = \begin{cases} g(x) & 0\leqslant x\leqslant 1 \\ 0 & x>1 \end{cases}$$

Moving the line of integration from $R(s)=c$ to $R(s)=c'<R(w)$, we obtain

$$\frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty} \frac{\Gamma\left(\frac{\mu}{2}+\frac{1}{4}+\frac{s}{2}\right)\Gamma\left(\frac{\nu}{2}+\frac{1}{4}+\frac{s}{2}\right)\chi(s)}{\Gamma\left(\frac{\mu}{2}-\frac{\alpha}{2}+\frac{1}{4}+\frac{s}{2}\right)\Gamma\left(\frac{\nu}{2}-\frac{\alpha}{2}+\frac{1}{4}+\frac{s}{2}\right)(s-\omega)}\,ds$$

$$+\frac{\Gamma\left(\frac{\mu}{2}+\frac{1}{4}+\frac{\omega}{2}\right)\Gamma\left(\frac{\nu}{2}+\frac{1}{4}+\frac{\omega}{2}\right)\chi(\omega)}{\Gamma\left(\frac{\mu}{2}-\frac{\alpha}{2}+\frac{1}{4}+\frac{\omega}{2}\right)\Gamma\left(\frac{\nu}{2}-\frac{\alpha}{2}+\frac{1}{4}+\frac{\omega}{2}\right)} = G(\alpha-\omega+1)$$

Consider now the integral

$$2\int_0^1 x^{2\rho-1}(1-x^2)^{\gamma-1}\ {}_2F_1(\alpha',\beta;\gamma;1-x^2)\ x^{s-1}\,dx$$

changing the variable of integration from x to ξ where $\xi = x^2$, we find from[15] that the integral has the value

$$\int_0^1 \xi^{\rho+\frac{s}{2}-\frac{3}{2}}(1-\xi)^{\gamma-1}\ {}_2F_1(\alpha',\beta;\gamma;1-\xi)\,d\xi$$
$$= \frac{\Gamma(\gamma)\,\Gamma\left(\rho+\frac{s}{2}-\frac{1}{2}\right)\Gamma\left(\gamma+\rho+\frac{s}{2}-\frac{1}{2}-\alpha'-\beta\right)}{\Gamma\left(\gamma+\rho+\frac{s}{2}-\frac{1}{2}-\alpha'\right)\Gamma\left(\gamma+\rho+\frac{s}{2}-\frac{1}{2}-\beta\right)}$$

It then follows as a result of applying the Mellin inversion formula that

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma\left(\rho+\frac{s}{2}-\frac{1}{2}\right)\Gamma\left(\gamma+\rho+\frac{s}{2}-\frac{1}{2}-\alpha'\,\beta\right)x^{-s}}{\Gamma\left(\gamma+\rho+\frac{s}{2}-\frac{1}{2}-\alpha'\right)\Gamma\left(\gamma+\rho+\frac{s}{2}-\frac{1}{2}-\beta\right)}\,ds$$

$$= \begin{cases} \dfrac{2}{\Gamma(\gamma)}\ x^{2\rho-1}(1-x^2)^{\gamma-1}\ {}_2F_1(\alpha',\beta;\gamma;1-x^2), & 0 \leqslant x \leqslant 1 \\ 0\ , & x > 0 \end{cases}$$

Putting $x = \frac{\xi}{\eta}$, $\rho = \frac{3}{4}+\frac{\mu}{2}-\frac{\alpha}{2}$, $\alpha' = \frac{\alpha}{2}$, $\beta = \frac{\mu}{2}-\frac{\nu}{2}+\frac{\alpha}{2}$, $\gamma = \alpha$ in this result, we obtain the integral

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma\left(\frac{\mu}{2}-\frac{\alpha}{2}+\frac{1}{4}+\frac{s}{2}\right)\Gamma\left(\frac{\nu}{2}-\frac{\alpha}{2}+\frac{1}{4}+\frac{s}{2}\right)}{\Gamma\left(\frac{\mu}{2}+\frac{1}{4}+\frac{s}{2}\right)\Gamma\left(\frac{\nu}{2}+\frac{1}{4}+\frac{s}{2}\right)}\left(\frac{\xi}{\eta}\right)^{-s}\,ds$$

$$= \begin{cases} \dfrac{2}{\Gamma(\alpha)}\left(\dfrac{\xi}{\eta}\right)^{\frac{1}{2}+\mu-\alpha}\left(1-\dfrac{\xi^2}{\eta^2}\right)^{\alpha-1}\ {}_2F_1\left(\dfrac{\alpha}{2},\ \dfrac{\alpha}{2}+\dfrac{\mu}{2}-\dfrac{\nu}{2};\ \alpha;\ 1-\dfrac{\xi^2}{\eta^2}\right), & \eta \geqslant \xi \\ 0\ , & 0 < \eta < \xi \end{cases}$$

so that

$$\chi(\omega) = \frac{2}{\Gamma(\alpha)}\int_0^1 g(\xi)\,\xi^{\mu+\frac{1}{2}}\,d\xi \int_0^1 \eta^{-\mu-\alpha-\omega+\frac{1}{2}}(\eta^2-\xi^2)^{\alpha-1}\times$$
$$\times\ {}_2F_1\left(\frac{\alpha}{2},\ \frac{\alpha}{2}+\frac{\mu}{2}-\frac{\nu}{2};\ \alpha;\ 1-\frac{\xi^2}{\eta^2}\right)d\eta,$$

changing the order of integration and substituting we find that

Now the integral occurring on the left-hand side of this equation is a regular function of ω for $R(\omega) > c'$. Therefore so also is the function

$$G(\alpha-\omega+1) - \frac{\Gamma\left(\frac{\mu}{2}+\frac{1}{4}+\frac{\omega}{2}\right)\Gamma\left(\frac{\nu}{2}+\frac{1}{4}+\frac{\omega}{2}\right)\chi(\omega)}{\Gamma\left(\frac{\mu}{2}-\frac{\alpha}{2}+\frac{1}{4}+\frac{\omega}{2}\right)\Gamma\left(\frac{\nu}{2}-\frac{\alpha}{2}+\frac{1}{4}+\frac{\omega}{2}\right)}$$

a regular function of ω for $R(\omega) > c'$. Hence so also

$$\chi(\omega) - \frac{\Gamma\left(\frac{\mu}{2}-\frac{\alpha}{2}+\frac{1}{2}+\frac{\omega}{2}\right)\Gamma\left(\frac{\nu}{2}-\frac{\alpha}{2}+\frac{1}{4}+\frac{\omega}{2}\right)}{\Gamma\left(\frac{\mu}{2}+\frac{1}{4}+\frac{\omega}{2}\right)\Gamma\left(\frac{\nu}{2}+\frac{1}{4}+\frac{\omega}{2}\right)} G(\alpha-\omega+1)$$

so that

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left[\chi(s) - \frac{\Gamma\left(\frac{\mu}{2}-\frac{\alpha}{2}+\frac{1}{4}+\frac{s}{2}\right)\Gamma\left(\frac{\nu}{2}-\frac{\alpha}{2}+\frac{1}{4}+\frac{s}{2}\right)}{\Gamma\left(\frac{\mu}{2}+\frac{1}{4}+\frac{s}{2}\right)\Gamma\left(\frac{\nu}{2}+\frac{1}{4}+\frac{s}{2}\right)} G(\alpha-s+1)\right]\frac{ds}{s-\omega} = 0$$

with $R(\omega) < c$.

Similarly from equation (3·2) we can show that

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\chi(s)\ \frac{ds}{s-\omega} = \chi(\omega),\ R(\omega) < c.$$

Thus

$$\chi(\omega) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma\left(\frac{\mu}{2}-\frac{\alpha}{2}+\frac{1}{4}+\frac{s}{2}\right)\Gamma\left(\frac{\nu}{2}-\frac{\alpha}{2}+\frac{1}{4}+\frac{s}{2}\right)}{\Gamma\left(\frac{\mu}{2}+\frac{1}{4}+\frac{s}{2}\right)\Gamma\left(\frac{\nu}{2}+\frac{1}{4}+\frac{s}{2}\right)(s-\omega)} G(\alpha-s+1)\,ds.$$

Now by definition

$$G(\alpha-s+1) = \int_0^1 g(\xi)\,\xi^{\alpha-s}\,d\xi$$

and we may write

$$\frac{1}{s-\omega} = \int_0^1 \eta^{s-\omega-1}\,d\eta$$

so that, interchanging the orders of integration, we obtain

$$\chi(\omega) = \int_0^1 g(\xi)\,\xi^{\alpha}\,d\xi \int_0^1 \eta^{-\omega-1}\,d\eta$$

$$\times \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma\left(\frac{\mu}{2}-\frac{\alpha}{2}+\frac{1}{4}+\frac{s}{2}\right)\Gamma\left(\frac{\nu}{2}-\frac{\alpha}{2}+\frac{1}{4}+\frac{s}{2}\right)}{\Gamma\left(\frac{\mu}{2}+\frac{1}{4}+\frac{s}{2}\right)\Gamma\left(\frac{\nu}{2}+\frac{1}{4}+\frac{s}{2}\right)}\left(\frac{\xi}{\eta}\right)^{-s} ds.$$

$$\chi(\omega) = \frac{2}{\Gamma(\alpha)}\int_0^1 \eta^{-\mu-\alpha-\omega+\frac{1}{2}}\, d\eta \int_0^\eta g(\xi)\, \xi^{\mu+\frac{1}{2}}\, (\eta^2-\xi^2)^{\alpha-1}$$

$$\times\, {}_2F_1\left(\frac{\alpha}{2}\,,\ \frac{\alpha}{2}+\frac{\mu}{2}-\frac{\nu}{2};\ \alpha\,;\ 1-\frac{\xi^2}{\eta^2}\right) d\xi$$

$$= \frac{2}{\Gamma(\alpha)}\int_0^1 \eta^{\alpha-\omega-1}\, d\eta \int_0^1 g(\eta z)\, z^{\mu+\frac{1}{2}}\, (1-z^2)^{\alpha-1} \times$$

$$\times\, {}_2F_1\left[\frac{\alpha}{2}\,,\ \frac{\alpha}{2}+\frac{\mu}{2}-\frac{\nu}{2};\ \alpha;\ (1-z^2)\,\eta^2\ \right] dz.$$

Now by the definition of $\chi(s)$ and the Mellin inversion formula

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)\, x^{-s}\, ds$$

we find for the function $f(x)$,

$$f(x) = \frac{1}{\Gamma(\alpha)}\int_0^1 \eta^{\alpha-1}\, d\eta \int_0^1 g(\eta z)\, z^{\mu+\frac{1}{2}}\, (1-z^2)^{\alpha-1} \times$$

$$\times\, {}_2F_1\left[\frac{\alpha}{2}\,,\ \frac{\alpha}{2}+\frac{\mu}{2}-\frac{\nu}{2}\ ;\ \alpha\ ;\ (1-z^2)\,\eta^2\right] dz$$

$$\times\, \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma\left(\frac{\mu}{2}+\frac{1}{4}+\frac{s}{2}\right)\Gamma\left(\frac{\nu}{2}+\frac{1}{4}+\frac{s}{2}\right)(\eta^2 x^2)^{-s/2}}{\Gamma\left(\frac{\mu}{2}+\frac{\alpha}{2}+\frac{3}{4}-\frac{s}{2}\right)\Gamma\left(\frac{\nu}{2}+\frac{\alpha}{2}+\frac{3}{4}-\frac{s}{2}\right)} 4^{s-\alpha}\, ds.$$

using Mellin inversion formula[13]

$$\frac{4^{-\alpha+\frac{1}{2}}}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma\left(\frac{\mu}{2}+\frac{1}{4}+\frac{s}{2}\right)\Gamma\left(\frac{\nu}{2}+\frac{1}{4}+\frac{s}{2}\right)}{\Gamma\left(\frac{\mu}{2}+\frac{\alpha}{2}+\frac{3}{4}-\frac{s}{2}\right)\Gamma\left(\frac{\nu}{2}+\frac{\alpha}{2}+\frac{3}{4}-\frac{s}{2}\right)}\left(\frac{\eta^2 x^2}{16}\right)^{-s/2} ds$$

$$= 4^{-\alpha+\frac{1}{2}}\, G^{2,0}_{0,4}\left(\frac{\eta^2 x^2}{16}\middle|\begin{matrix}\frac{\mu}{2}+\frac{1}{4}\ ;\ \frac{\nu}{2}+\frac{1}{4}\ ,\ \frac{1}{4}-\frac{\nu}{2}-\frac{\alpha}{2}\ ,\ \frac{1}{4}-\frac{\mu}{2}-\frac{\alpha}{2}\end{matrix}\right)$$

which gives finally

$$(3\cdot3)\qquad f(x) = \frac{2^{-2\alpha}\sqrt{x}}{\Gamma(\alpha)}\int_0^1 \eta^{\alpha+\frac{1}{2}}\, G^{2,0}_{0,4}\left(\frac{\eta^2 x^2}{16}\middle|\begin{matrix}\frac{\mu}{2}\ ,\ \frac{\nu}{2}, -\frac{\mu}{2}-\frac{\alpha}{2}, -\frac{\nu}{2}-\frac{\alpha}{2}\end{matrix}\right) d\eta$$

$$\times \int_0^1 g(\eta z)\, z^{\mu+\frac{1}{2}}\, (1-z^2)^{\alpha-1}\ \ {}_2F_1\left[\frac{\alpha}{2}\,,\ \frac{\alpha}{2}+\frac{\mu}{2}-\frac{\nu}{2}\ ;\ \alpha\ ;\ (1-z^2)\,\eta^2\ \right] dz.$$

This solution is valid for $\alpha > 0$.

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References

1. Titchmarsh, E. C. Introduction to the theory of Fourier Integrals, 334–339, 1937.
2. Busbridge, I. W. Dual Integral equations. *Proc. Lond. Math. Soc.*, **44**, 115–129, 1938.
3. Gordon, A. N. Dual Integral equations. *J. Lond. Math. Soc.*, **29**, 360–363, 1954.
4. Copson, E. T. On certain Dual Integral equations. *Proc. Glas. Math. Ass.*, **5**, 21–24, 1961.
5. Sneddon, I. N. The elementary solution of dual Integral equation. *Proc. Glas. Math. Asso.*, **4**, 108–110, 1960.
6. Tranter, C. J. On some dual Integral equations. *Quart. J. Math. Oxford, Ser*, (**2**)2, 60-66, 1951.
7. Noble, B. Certain dual Integral equations. *J. Math. Phys.*, **37**, 128-136, 1958.
8. Williams, W. E. The solution of certain dual Integral equations. *Proc. Edinburgh Math. Soc.*, **12**, 213–216, 1961.
9. Lowengrub, M. and Sneddon, I. N. The solution of a pair of Dual Integral equations. *Proc. Glas. Math. Asso.*, **6**(1), 14–18, 1963.
10. Watson, G. N. *Quart. J. of Math. (Oxford)*, **2**, 298-309, 1931.
11. Bhatnagar, K. P. Thesis for the Ph.D. degree of Lucknow University, Chapter IV.
12. Narain, N. G-functions as unsymmetrical Fourier Kernels I. *Proc. Amer. Math. Soc.* **13**(6), 950–959, Dec. 1962.
13. Erdelyi, A. Tables of Integral Transforms. *McGraw Hill*, **1**(34), 353, 1954.
14. Sneddon, I. N. Fourier Transforms. *McGraw Hill*, 43, 1951.
15. Erdelyi, A. Tables of Integral transforms *McGraw Hill*, **2**(4), 399, 1954.

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The use of Bessel function and Jacobi polynomial in the cooling of a heated cylinder

By

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Abstract

In this paper we have employed the Bessel function of the first kind and the Jacobi polynomial to obtain the solutions of the fundamental differential equation of the cooling of a heated cylinder.

1. Introduction: Cases in which heat is produced in solids are becoming increasingly important in technical applications [4, pp. 11-12]. Space research and Nuclear reactors also give rise to different problems of heat transfer.

In this paper we consider the problem of the cooling of an infinitely long cylinder of radius a, heated to the temperature $u_0 = f(r)$ [r is the distance from the axis] and radiating heat into the surrounding medium at zero temperature. From a mathematical point of view, the problem reduces to solving the equation of heat conduction [2, p. 155, (6·7·1)]

$$c\,\rho\,\frac{\partial u}{\partial t} = k\,\nabla^2 u, \tag{1·1}$$

Subject to the boundary condition

$$\left(-\frac{\partial u}{\partial t} + h\,u\right)\Big|_{r=a} = 0, \tag{1·2}$$

and the initial condition

$$u/_{t} = u_0 = f(r), \tag{1·3}$$

where the object have thermal conductivity k, heat capacity c, density ρ and emissivity λ and $h = \lambda/k$.

In this paper we shall consider the following values of the temperature

$$f(r) = \{1 - r^2/a^2\}^{\nu/2} J_\nu\{\lambda\sqrt{(1 - r^2/a^2)}\}, \tag{1·4}$$

and

$$f(r) = \{1 - r^2/a^2\}^{\beta} P_\nu^{(0,\beta)}\{1 - 2r^2/a^2\}. \tag{1·5}$$

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Any function which is bounded and has a finite number of maxima and minima can be represented by a series of Bessel functions; hence the temperature source (1·4) is of general character and of great physical interest. In view of the expansion property of Jacobi polynomials [5, p. 24], the temperature (1·5) may encompass several cases of interest.

The following formulae are required in the proof:

$$(1\cdot6)\qquad \int_0^a r\{1-r^2/a^2\}^{\nu/2} J_\nu\{\lambda\sqrt{(1-r^2/a^2)}\}\, J_0(x_m r/a)\, dr$$

$$= \frac{a^2\lambda^\nu}{\{\lambda^2+x^2_m\}^{(\nu+1)/2}} J_{\nu+1}\{\sqrt{(\lambda^2+x^2_m)}\},\ Re\ \nu > -1,$$

which follows from [3, p. 299, (26)].

$$(1\cdot7)\qquad \int_0^a r\{1-r^2/a^2\}^\beta P_\nu^{(0,\beta)}\{1-2r^2/a^2\} J_0(x_m r/a)\, dr$$

$$= \frac{2^\beta a^2 \sqrt{(\beta+\nu+1)}}{\nu!\, x_m{}^{\beta+1}} J_{\beta+2\nu+1}(x_m),\ Re\ \beta > -1,$$

which follows from [1, p. 189].

2. *Solution of the problem*:

The solutions of (1·1) to be obtained are

$$(2\cdot1)\qquad u(r,t) = 2\lambda^\nu \sum_{n=1}^{\infty} \frac{J_{\nu+1}\{\sqrt{(\lambda^2+x_n{}^2)}\} J_0(x_n r/a)\, e^{-x^2_n\, t/a^2\, b}}{\{\lambda^2+x_n{}^2\}^{(\nu+1)/2}\,[J_0{}^2(x_n)+J_1{}^2(x_n)]},\ Re\ \nu > -1.$$

$$(2\cdot2)\qquad u(r,t) = \frac{2^{\beta+1}\Gamma(\beta+\nu+1)}{\nu!} \sum_{n=1}^{\infty} \frac{J_{\beta+2\nu+1}(x_n)\, J_0(x_n r/a)\, e^{-x^2_n t/a^2 b}}{x_n{}^{\beta+1}\,[J_0{}^2(x_n)+J_1{}^2(x_n)]},\ Re\ \beta > -1.$$

Proof: The solution of (1·1) as given in [2, p. 155, (6·7·6)] is

$$(2\cdot3)\qquad u(r,t) = \sum_{n=1}^{\infty} M_n J_0(x_n r/a)\, e^{-x^2_n t/a^2 b}, \quad \left(b = \frac{c\rho}{k}\right)$$

where because of the initial condition (1·3), the coefficients M_n must be chosen to satisfy the relation

$$(2\cdot4)\qquad f(r) = \sum_{n=1}^{\infty} M_n J_0(x_n r/a),\ 0 \leqslant r < a.$$

By virtue of (1·4) we have

$$(2\cdot5)\qquad f(r) = \{1-r^2/a^2\}^{\nu/2} J_\nu\{\lambda\sqrt{(1-r^2/a^2)}\} = \sum_{n=1}^{\infty} M_n J_0(x_n r/a),\ 0 \leqslant r < a.$$

Multiplying both sides of (2·5) by $r J_0(x_m r/a)$ and integrating with respect to r from 0 to a, we get

$$(2\cdot6)\qquad \int_0^a r\{1-r^2/a^2\}^{\nu} J_\nu\{\lambda\sqrt{(1-r^2/a^2)}\} J_0(x_m r/a)\, dr$$

$$= \sum_{n=1}^{\infty} M_n \int_0^a r J_0(x_n r/a) J_0(x_m r/a)\, dr.$$

Now using (1·6) and the orthogonality property of Bessel functions [2, p. 130, (5·14·9)]*, we obtain

$$(2\cdot7)\qquad M_m = \frac{2\lambda^{\nu} J_{\nu+1}\{\sqrt{(\lambda^2 + x_m^2)}\}}{\{\lambda^2 + x_m^2\}^{(\nu+1)/2}[J_0'^2(x_m) + J_0^2(x_m)]}.$$

Now using the relation $J'_0(z) = -J_1(z)$, the solution (2·1) follows from (2·3) and (2·7) immediately.

The solution (2·2) can be obtained similarly by virtue of the relations (1·5) and (1·7).

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References

1. Bhonsle, B. R. On some results involving Jacobi polynomials. *Journ. Ind. Math. Soc.*, **26**, 187–190, 1962.
2. Lebedev, N. N. Special functions and their applications. *Prentice Hall, Inc, Englewood Cliffs, N.J.*, 1965.
3. Luke, Y. L. Integrals of Bessel functions. *McGraw-Hill, New York*, 1962.
4. Carslaw, H. S. and Jaeger, J. C. Conduction of heat in solids, 2nd Ed., *Oxford University Press, London*, 1959.
5. Szego, G. Orthogonal polynomials. *American Mathematical Society*, 1959.

$$*\int_0^a r J_\nu(x_{\nu m} r/a) J_\nu(x_{\nu n} r/a)\, dr = \begin{cases} 0, \text{ if } m \neq n, \\ a^2/2\,[J'_\nu{}^2(x_{\nu n}) + (1-\nu^2/x^2_{\nu n}) \\ \qquad J_\nu^2(x_{\nu n})], \text{ if } m = n. \end{cases}$$

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On certain integral equations involving Hypergeometric and incomplete Gamma functions

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Abstract

The solutions of three integral equations involving hypergeometric and incomplete Gamma functions have been established in this paper. One of the special cases is a known result given recently by K. C. Rusia.

1. Introduction

In the solution of certain boundary value problems we come across the integral equations of the type

$$g(x) = \int_a^b F(xu)\, f(u)\, du,$$

where $g(x)$ and $F(xu)$ are known functions and $f(x)$ is to determined.

In a recent paper, Jet Wimp [10, p. 42] established an integral equation pair where the kernel of each integral involves the Gaussian hypergeometric function. R. G. Buschman [1, 2, 3, 4] has given solutions for a number of integral equations involving Legendre, Chebyshev polynomials etc. A. Erdelyi [7, p. 15] applied the theory of fractional integration to the solution of an integral equation involving Legendre function.

Here using the method of Laplace transform, the solutions of certain integral equations involving hypergeometric and incomplete Gamma functions have been established. Several special cases of interest have also been derived.

If

$$F(p) = \int_0^\infty e^{-pt} f(t)\, dt, \qquad Re\, p > 0 \tag{1·1}$$

then $F(p)$ is called the Laplace transform of $f(t)$ and is denoted by

$$F(p) = L[f(t)] \quad \text{or} \quad F(p) \doteqdot f(t) \tag{1·2}$$

An integral equation of the form

$$\int_0^t \phi(t-u)\, g(u)\, du = f(t) \tag{1·3}$$

has the solution of the type

(1·4) $$\int_0^t \psi(t-u) f(u)\, du = g(t),$$

provided the inverse Laplace transform of $\left[\dfrac{\text{Laplace transform of } f(t)}{\text{Laplace transform of } \phi(t)}\right]$ exists.

2. The following results are required in establishing the theorems of this paper.

(2·1) $$(p^2 + a^2)^{-n-1} \doteqdot \frac{\pi^{\frac{1}{2}}\, t^{n+\frac{1}{2}}}{(2a)^{n+\frac{1}{2}}\, \Gamma(n+1)} J_{n+\frac{1}{2}}(at), \quad Re\, n > -1$$ [5, p. 182]

(2·2) $$(p+b)^{-m} \doteqdot \frac{t^{m-1} e^{-bt}}{\Gamma(m)}, \quad Re\, m > 0, \quad Re\, p > -Re\, b$$ [5, p. 144]

(2·3) If $f_1(t) \doteqdot F_1(p)$ and $f_2(t) \doteqdot F_2(p)$, then

$$\int_0^t f_1(u) f_2(t-u)\, du \doteqdot F_1(p)\, F_2(p)$$ [5, p. 131]

(2·4) $$\Gamma(l)\, p^{a-l} (p-\alpha)^{-a} \doteqdot t^{l-1}\, {}_1F_1(a\,;\, l\,;\, \alpha t),$$

$$Re\, l > 0, \quad Re\, p > 0, \quad Re\, \alpha > 0$$ [5, p. 217]

(2·5) If $\phi(p) \doteqdot f(t)$, then $\phi(p+\beta) \doteqdot e^{-\beta t} f(t)$. [5, p. 129]

From (2·4) and (2·5), we get

(2·6) $$\frac{\Gamma(\lambda+\nu+2)}{(p-\beta)^{\lambda+\mu+2}} \frac{(p+\alpha-\beta)^{\mu-\nu}}{} \doteqdot t^{\lambda+\nu+1}\, e^{(\beta-\alpha)t}\, {}_1F_1(\lambda+\mu+2;\ \lambda+\nu+2\ ;\ \alpha t),$$

$$Re(\lambda+\nu+2) > 0, \quad Re(p+\alpha-\beta) > 0, \quad Re\, \alpha > 0$$

(2·7) $p^n F(p) \doteqdot f^{(n)}(t)$ provided that $f(0) = f'(0) = \ldots = f^{n-1}(0) = 0$ [5, p. 129]

(2·8) $$\frac{\Gamma(\alpha+n+1)}{n!} \cdot \frac{(p-k-\lambda)^n}{(p-\lambda)^{\alpha+n+1}} \doteqdot t^{\alpha}\, e^{\lambda t}\, L_n^{\alpha}(kt),$$

$$Re\, \alpha > -1, \quad Re\,(p-\lambda) > 0$$ [5, p. 175]

(2·9) $$\Gamma(\nu)\, \alpha^{\nu} (p-\beta)^{-1} (p+\alpha-\beta)^{-\nu} \doteqdot e^{\beta t}\, \gamma(\nu, \alpha t),$$

$$Re\, \nu > -1, \quad Re\, p > Re\, \beta$$ [5, p. 179]

(2·10) $$(p+\alpha)^{-m} (p+\beta)^{-n} (p+\gamma)^{-l} \doteqdot \frac{t^{l+m+n-1} e^{-\gamma t}}{\Gamma(m+n)} \times$$

$$\sum_{r=0}^{\infty} \frac{(\gamma-\beta)^r\, t^r\, \Gamma(m+n+r)}{r!\, \Gamma(l+m+n+r)}\, {}_2F_2\left(\begin{matrix} m+n+r,\ m \\ l+m+n+r,\ m+n; \end{matrix} (\beta-\alpha)t\right),$$

$$Re\, m > 0, \quad Re\, n > 0$$ [8, p. 41]

3. *Theorem* 1 : If

(*i*) $m = 1, 2, 3, \ldots; n = 0, 1, 2, \ldots$.

(*ii*) $f^{2n+m+2}(t)$ is continuous for $0 \leqslant t < \infty$ and

$f(0) = f'(0) = \ldots\ldots = f^{2n+m+1}(0) = 0$

then the integral equation

$$(3\cdot1)\qquad \sum_{r=0}^{\infty} \frac{b^r (2n+2)_r}{r\,!\,\Gamma(2n+m+r+2)} \int_0^t e^{-b(t-u)}\,(t-u)^{2n+m+r+1} \times$$

$$ {}_2F_3\left[\begin{matrix} n+r/2+1,\ n+r/2+3/2 \\ n+3/2,\ n+m/2+r/2+1,\ n+m/2+r/2+3/2; \end{matrix} -\frac{a^2}{4}\,(t-u)^2\right] g(u)\,du = f(t)$$

has the solution

$$(3\cdot2)\qquad g(t) = (D^2 + a^2)^{n+1}\,(D+b)^m f(t), \quad \text{where } D \equiv d/dt$$

provided that the left hand side of (3·1) exists.

Proof: Applying the convolution theorem (2·3) to (2·1) and (2·2), we get

$$(3\cdot3)\qquad (p^2+a^2)^{-n-1}\,(p+b)^{-m} \doteqdot \frac{\pi^{\frac{1}{2}}\,e^{-bt}}{(2a)^{n+\frac{1}{2}}\,\Gamma(n+1)\,\Gamma(m)} \int_0^t u^{n+\frac{1}{2}}\,e^{bu}\,J_{n+\frac{1}{2}}(au)(t-u)^{m-1}\,du$$

Expanding the exponential function e^{bu} in powers of u, we get

$$(3\cdot4)\qquad (p^2+a^2)^{-n-1}\,(p+b)^{-m} \doteqdot \frac{\pi^{\frac{1}{2}}\,e^{-bt}}{(2a)^{n+\frac{1}{2}}\,\Gamma(n+1)\,\Gamma(m)} \times$$

$$\sum_{r=0}^{\infty} \int_0^t \frac{b^r}{r\,!}\,u^{n+r+\frac{1}{2}}\ J_{n+\frac{1}{2}}(au)\,(t-u)^{m-1}\,du.$$

By virtue of relation [6, p. 193 (56)]

$$(3\cdot5)\qquad \frac{1}{\Gamma(\mu)} \int_0^t u^{\lambda-1}\,J_\nu(au)\,(t-u)^{\mu-1}\,du = \frac{\Gamma(\lambda+\nu)}{\Gamma(\nu+1)\,\Gamma(\lambda+\mu+\nu)}\,(\tfrac{1}{2}a)^\nu\,t^{\lambda+\mu+\nu-1} \times$$

$${}_2F_3\left(\begin{matrix} \frac{\lambda+\nu}{2},\ \frac{\lambda+\nu+1}{2} \\ \nu+1,\ \frac{\lambda+\mu+\nu}{2},\ \frac{\lambda+\mu+\nu+1}{2}; \end{matrix} \frac{-a^2t^2}{4}\right), \quad Re\ \mu > 0,\ Re\ (\lambda+\nu) > 0,$$

we get

$$(3\cdot6)\qquad (p^2+a^2)^{-n-1}\,(p+b)^{-m} \doteqdot \sum_{r=0}^{\infty} \frac{b^r\,e^{-bt}\,(2n+2)_r\,t^{2n+m+r+1}}{r\,!\,\Gamma(2\,n+m+r+2)} \times$$

$${}_2F_3\left(\begin{matrix} n+r/2+1,\ n+r/2+3/2 \\ n+3/2,\ n+m/2+r/2+1,\ n+m/2+r/2+3/2; \end{matrix} \frac{-a^2t^2}{4}\right), \quad Re\,m>0,\ Re(2n+r+2)>0$$

Applying the convolution theorem (2·3) to (3·1), we get

$$L\left[\sum_{r=0}^{\infty} \frac{b^r\,(2n+2)_r\ e^{-bt}\,t^{2n+m+r+1}}{r\,!\,\Gamma(2\,n+m+r+2)} \times\right.$$

$$\left. {}_2F_3\left(\begin{matrix} n+r/2+1,\ n+r/2+3/2 \\ n+3/2,\ n+m/2+r/2+1,\ n+m/2+r/2+3/2; \end{matrix} \frac{-a^2t^2}{4}\right)\right] G(p) = F(p)$$

Using (3·6), we get

(3·7) $$G(p) = (p^2+a^2)^{n+1}\,(p+b)^m\,F(p)$$

Taking inverses, we obtain the desired result (3·2).

Special Case : In (3·1), put $b = 0$, $m = 2l$ and replace n by $n-1$ then the integral equation

(3·8) $$\int_0^t (t-u)^{2l+2n-1}\,{}_1F_2[n\,;\ l+n,\ l+n+\tfrac{1}{2}\,;\ -\tfrac{1}{4}a^2\,(t-u)^2]\,g(u)\,du = f(t)$$ has the solution

$$g(t) = \frac{D^{2l}\,(D^2+a^2)^n f(t)}{\Gamma(2l+2n)}, \text{ where } D \equiv \frac{d}{dt}$$

Theorem 2 : If

(*i*) $l = 0, 1, 2, 3, \ldots\,; m, n = 1, 2, 3, \ldots$

(*ii*) $f^{l+m+n}(t)$ is continuous for $0 \leqslant t < \infty$ and

$f(0) = f'(0) = \ldots\ldots = f^{l+m+n-1}(0) = 0$,

then the integral equation

(3·9) $$\sum_{r=0}^{\infty} \frac{\Gamma(m+n+r)\,(\gamma-\beta)^r}{\Gamma(l+m+n+r)\,r\,!} \int_0^t e^{-\gamma(t-u)}\,(t-u)^{l+m+n+r-1} \times$$

$$ {}_2F_2\left[\begin{matrix} m+n+r,\ m \\ l+m+n+r,\ m+n\,; \end{matrix}\ (\beta-\alpha)\,(t-u)\right] g(u)\,du = f(t)$$

has the solution

(3·10) $$g(t) = \frac{(D+\alpha)^m\,(D+\beta)^n\,(D+\gamma)^l}{\Gamma(m+n)}\,f(t), \quad \text{where } D \equiv \frac{d}{dt}$$

provided that the left hand side of (3·9) exists.

Proof : Applying the convolution theorem (2·3) to (3·9), we get

$$L\left\{\sum_{r=0}^{\infty} \frac{\Gamma(m+n+r)\,(\gamma-\beta)^r}{\Gamma(l+m+n+r)\,r\,!}\,e^{-\gamma t}\,t^{l+m+n+r-1} \times\right.$$

$$\left. {}_2F_2\left[\begin{matrix} m+n+r,\ m \\ l+m+n+r,\ m+n\,; \end{matrix}\ (\beta-\alpha)\,t\right]\right\} G(p) = F(p)$$

By virtue of (2·10), we get

(3·11) $$G(p) = \frac{(p+\alpha)^m\,(p+\beta)^n\,(p+\gamma)^l}{\Gamma(m+n)}\,F(p)$$

Taking inverses, we obtain the required result (3·10)

Special Case : In (3·9), put $l = 0$, $n = 1$, $m = \nu$ and replace α by $a - b$, γ and β by $-b$

then the integral equation

(3·12) $$\int_0^t e^{b(t-u)}\,\gamma[\nu,\ a(t-u)\,]\,g(u)\,du = f(t)$$

has the solution

$$g(t) = \frac{(D+a-b)^{\nu}\,(D-b)}{\Gamma(\nu)\,a^{\nu}}\, f(t), \text{ where } D \equiv \frac{d}{dt}. \tag{3·13}$$

Further if we put $b = 0$, we obtain a result given recently by Rusia [9, p. 173]

Theorem 3 : If

(*i*) ν is a positive integer

(*ii*) $f^{\nu+2}(t)$ is continuous for $0 \leq t < \infty$ and

$$f(0) = f'(0) = \ldots\ldots = f^{\nu+1}(0) = 0$$

then the integral equation

$$\int_0^t e^{\beta(t-u)}\, \gamma[\nu, \alpha(t-u)]\, g(u)\, du = f(t) \tag{3·14}$$

has the solution

$$g(t) = \frac{1}{\alpha^{\nu}\,\Gamma(\nu)} \int_0^t e^{\beta(t-u)}\, L_{\nu}[\alpha\,(u-t)]\, [(D-\beta)^{\nu+2} f(u)]\, du, \tag{3·15}$$

where $D \equiv \frac{d}{du}$

Proof : Applying the convolution theorem (2·3) to (3·14), we get

$$G(p)\; L[e^{\beta t}\, \gamma(\nu, \alpha t)] = F(p) \tag{3·16}$$

Using (2·9), we obtain

$$G(p) = \frac{(p+\alpha-\beta)^{\nu}}{\Gamma(\nu)\,\alpha^{\nu}\,(p-\beta)^{\nu+1}}\,(p-\beta)^{\nu+2} F(p), \tag{3·17}$$

Applying (2·7) and (2·8), we get

$$G(p) = \frac{1}{\alpha^{\nu}\,\Gamma(\nu)}\, L[e^{\beta t}\, L_{\nu}(-\alpha t)].\; L[(D-\beta)^{\nu+2} f(t)], \text{ where } D \equiv \frac{d}{dt} \tag{3·18}$$

Applying the convolution theorem (2·3) to (3·18) we obtain the desired result (3·15)

4. *Application* : In this section we make use of theorem 3 to evaluate the integral

$$\int_0^t e^{-\alpha u}\, L_{\nu}[\alpha\,(u-t)]\, [\,(D-\alpha)^{\nu+2}\{\, u^{\lambda+\nu+1}\, {}_1F_1(\lambda+\mu+2;\, \lambda+\nu+2;\, \alpha u)\,\}\,]\, du \tag{4·1}$$

$$= \frac{\Gamma(\mu+1)\,\Gamma(\lambda+\nu+2)}{\Gamma(\lambda+\mu+1)}\; t^{\lambda}\, L_{\mu}^{\lambda}(-\alpha t)$$

where $R(\lambda+1) > 0$, $R(\mu+1) > 0$ and $\nu = 0, 1, 2, \ldots\ldots$

Proof : We have the following identity :

$$\Gamma(\nu)\,\alpha^{\nu}\,(p-\beta)^{-1}\,(p+\alpha-\beta)^{-\nu} \times \frac{(p+\alpha-\beta)^{\mu}\,\Gamma(\mu+\lambda+1)}{(p-\beta)^{\lambda+\mu+1}\,\Gamma(\mu+1)} \tag{4·2}$$

$$= \frac{\Gamma(\nu)\,\alpha^{\nu}\,\Gamma(\lambda+\mu+1)}{\Gamma(\lambda+\nu+2)\,\Gamma(\mu+1)} \times \frac{\Gamma(\lambda+\nu+2)\,(p+\alpha-\beta)^{\mu-\nu}}{(p-\beta)^{\lambda+\mu+2}}$$

Applying (2·8), (2·9) and (2·6) to (4·2), we get

$$(4\cdot3)\qquad L\,[e^{\beta t}\,\gamma(\nu,\,\alpha t)]\times L\,[t^{\lambda}\,e^{\beta t}\,L_{\mu}^{\lambda}\,(-\alpha t)\,]$$
$$=\frac{\alpha^{\nu}\,\Gamma(\nu)\,\Gamma(\lambda+\mu+1)}{\Gamma(\lambda+\nu+2)\,\Gamma(\mu+1)}\,L\,[t^{\lambda+\nu+1}\,e^{(\beta-\alpha)t}\,{}_1F_1(\lambda+\mu+2\,;\,\lambda+\nu+2\,;\,\alpha t)],$$

where $Re(\lambda+1)>0$, $Re(\nu+1)>0$, $Re\,\alpha>0$, $Re(p-\beta)>0$

Applying the convolution theorem (2·3) to (4·3), we get

$$(4\cdot4)\qquad \int_0^t e^{\beta(t-u)}\,\gamma\,[\nu,\,\alpha\,(t-u)\,]\,[u^{\lambda}\,e^{\beta u}\,L_{\mu}^{\lambda}\,(-\alpha u)\,]\,du$$
$$=\frac{\alpha^{\nu}\,\Gamma(\nu)\,\Gamma(\lambda+\mu+1)}{\Gamma(\lambda+\nu+2)\,\Gamma(\mu+1)}\,[t^{\lambda+\nu+1}\,e^{(\beta-\alpha)t}\,{}_1F_1(\lambda+\mu+2\,;\,\lambda+\nu+2\,;\,\alpha t)]$$

Comparing the result (4·4) with (3·14), we find that

$$f(t)=\frac{\alpha^{\nu}\,\Gamma(\nu)\,\Gamma(\lambda+\mu+1)}{\Gamma(\lambda+\nu+2)\,\Gamma(\mu+1)}\,[t^{\lambda+\nu+1}\,e^{(\beta-\alpha)t}\,{}_1F_1\,(\lambda+\mu+2\,;\,\lambda+\nu+2\,;\,\alpha t)\,]$$

and

$$g(t)=t^{\lambda}\,e^{\beta t}\,L_{\mu}^{\lambda}\,(-\alpha t)$$

Substituting the value of $f(t)$ and $g(t)$ in (3·15) and simplifying, we obtain the desired result (4·1).

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References

1. Buschman, R. G. An inversion integral for a Legendre transformation. *Amer. Math. Monthly*, **69**(4), 288–289, 1962.
2. ——— An inversion integral. *Proc. Amer. Math. Soc.*, **13**(5) 675–677, 1962.
3. ——— An inversion integral for a general Legendre transformation. *Siam Review*, **5**(3), 232–233, 1963.
4. ——— Convolution equation with generalised Leguerre polynomial kernel. *Siam Review*, **6**(2), 166–167, 1964.
5. Erdelyi, A. Tables of integral transforms. **Vol. I**, *Mc–Graw-Hill, New York*, 1954.
6. ——— Tables of integral transforms. **Vol. II**, *Mc–Graw Hill, New York*, 1954.
7. ——— An integral equation involving Legendre function. *J. Soc. Indust. Appl. Math.*, **12**(1), 15–30, 1964.
8. Kulshreshtha, S. K. A Theorem on Laplace transform. *Studies in Mathematics, University of Rajasthan, Jaipur*, 39–47, 1965.
9. Rusia, K. C. A study on certain integral equations with application to problem of Engineering. *Thesis approved by the University of Jabalpur*, 1967.
10. Wimp, Jet. Two integral transform pair involving hypergeometric function. *Proc. Glasgow, Math. Assoc.*, **7**, 42–44, 1965.

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On Distribution and Frequency Functions of Several Variables

By

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Abstract

In our earlier paper[1] we defined Dirac's δ-function and Green's G-function in several variables and studied their application to the Convolution transform of functions of a number of variables, as defined by Jain[2], on the lines of Convolution transform of functions of a single variable[3].

The purpose of the present note is to extend the concept of the statistical terms so as to be applicable to the problems of convolution of functions of several variables.

We, here, define Stieltjes Convolution in several variables, and also establish, theorems on *Mean* and *Variance* of Stieltjes Convolution of functions of two variables. Similar theorems in case of one variable have established by Widder [4 ; 257].

1. Hirschman and Widder, [3 ; p. 4], define the Lebesgue Convolution Transform* as

$$f(x) = \int_{-\infty}^{\infty} G(x - t)\, \phi(t)\, dt \tag{1·1}$$

Jain[2], extended this concept and defined

$$f(x_1, x_2, \ldots, x_n) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} G(x_1 - t_1, \ldots, x_n - t_n)\, \phi(t_1, t_2, \ldots, t_n) \prod_{r=1}^{n} dt_r \tag{1·2}$$

2. Hirschman and Widder, [3, p. 22], define Stieltjes Convolution Transform # as

$$f(x) = \int_{-\infty}^{\infty} G(x - t)\, d\phi(t) \tag{2·1}$$

and we, in this note, extend this to the case of n-variables as

$$F(x_1, x_2, \ldots, x_n) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} G(x_1 - t_1, \ldots, x_n - t_n) \prod_{r=1}^{n} dt_r\, \phi(t, \ldots, t_n)$$

$$(2\cdot2) \qquad = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} G(x_1 - t_1, \ldots, x_n - t_n) \prod_{r=1}^{n} \left(\frac{\partial \phi}{\partial t_r} dt_r \right)$$

We have, on taking $n = 2$, (2·2) reducing to

$$(2\cdot3) \qquad f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(x - u, y - v)\, d_u \phi(u, v)\, d_v \phi(u, v)$$

where $d_x f(x, y)$ stands for the partial differrntial of $f(x, y)$ with regard to x.

(2·3) represents the Stieltjes Convolution Transform in two variables.

3. A function $\alpha\ (t_1, t_2)$ defined in $-\infty < t_1, t_2 < \infty$ is a *Distribution Function* if and only if :

(A) $\alpha(t_1, t_2) \in \uparrow$ with regard to t_1 (t_2 treated as constant)

or with regard to t_2 (t_1 treated as constant)

(B) $\underset{t_1 \to -\infty}{Lt}\ \alpha\ (t_1, t_2) = 0 = \underset{t_2 \to \infty}{Lt}\ \alpha(t_1, t_2)$

(C) if $\underset{t_1 \to \infty}{Lt}\ \alpha(t_1, t_2) = \alpha_1(t_2)$ and $\underset{t_2 \to \infty}{Lt}\ \alpha(t_1, t_2) = \alpha_2\ (t_1)$

then $\underset{t_2 \to \infty}{Lt}\ \alpha_1(t_2) = \underset{t_1 \to \infty}{Lt}\ \alpha_2(t_2) = 1$

A distribution function α is *normalized* if

$$\alpha(t_1, t_2) = \frac{\alpha(t_1^+, t_2) + \alpha(t_1^-, t_2)}{2}$$

$$= \frac{\alpha(t_1, t_2^+) + \alpha(t_1, t_2^-)}{2}$$

$$= \frac{\alpha(t_1^+, t_2) + \alpha(t_1, t_2^+) + \alpha(t_1^-, t_2) + \alpha(t_1, t_2^-)}{4}$$

4. The *mean* m_α of a distribution function $\alpha(t_1, t_2)$ is the integral

$$m_\alpha = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} t_1\, d_{t_1}\, \alpha(t_1, t_2)\, t_2\, d_{t_2}\, \alpha(t_1, t_2)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} t_1 t_2 \frac{\partial \alpha}{\partial t_1} \frac{\partial \alpha}{\partial t_2} dt_1\, dt_2$$

where the notation $d_{t_1}\ \alpha(t_1, t_2)$ represents the partial differential of α with respect to t_1, so that $d_{t_1} \alpha(t_1, t_2) = \frac{\partial \alpha}{\partial t_1} dt_1$. A similar meaning is assigned to $d_{t_2}\ \alpha(t_1, t_2)$. It is useful to define here the *partial means* $m^{\alpha}{}_{t_1}\ (t_2)$ and $m^{\alpha}{}_{t_2}\ (t_1)$ which are functions of t_2 and t_1 respectively. Their usefulness is marked in defining the statistical term *Variance* in terms of the partial means.

5. The *partial means* $m^{\alpha}_{t_1}$ and $m^{\alpha}_{t_2}(t_1)$ are the integrals

$$m^{\alpha}_{t_1}(t_2) = \int_{-\infty}^{\infty} t_1 \, d_{t_1} \alpha(t_1, t_2)$$

$$= \int_{-\infty}^{\infty} t_1 \frac{\partial \alpha}{\partial t_1} dt_1$$

$$m^{\alpha}_{t_2}(t_1) = \int_{-\infty}^{\infty} t_2 \, d_{t_2} \alpha(t_1, t_2) = \int_{-\infty}^{\infty} t_2 \frac{\partial \alpha}{\partial t_2} dt_2$$

provided these integrals converage.

Physical interpretations :

Taking $\alpha(t_1, t_2)$ as defining a distribution of unit mass in the (t_1, t_2) plane $\alpha(b, t_2) - \alpha(a, t_2)$ represents units of mass on the line t_2 = constant between $t_1 = a$ and $t_1 = b$. Similarly $\alpha(t_1, b') - \alpha(t_1, a')$ represents units of mass on line t_1 = constant in the interval $(t_2 = a', t_2 = b')$.

$m^{\alpha}_{t_2}(t_1)$ gives the t_1 - coordinate of the centre of gravity of the mass distributed on the line t_2 = constant. Similarly $m^{\alpha}_{t_1}(t_2)$ gives the t_2 - coordinate of the centre of gravity of the mass distributed on the line t_1 = constant.

6. The *Variance* V_{α} of a distribution function $\alpha(t_1, t_2)$ is the integral

$$V_{\alpha} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(t_1 - m^{\alpha}_{t_1}(t_2) \right)^2 \left(t_2 - m^{\alpha}_{t_2}(t_1) \right)^2 d^{\alpha}_{t_1} \alpha(t_1, t_2) \; d^{\alpha}_{t_2}(t_1, t_2).$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(t_1 - m^{\alpha}_{t_1} \right)^2 \left(t_2 - m^{\alpha}_{t_2} \right)^2 \frac{\partial \alpha}{\partial t_1} \frac{\partial \alpha}{\partial t_2} dt_1 \, dt_2$$

7. Consider distribution functions which are absolutely continuous. The derivative of an absolutely continuous distribution function is defined as a *Frequency Function.*

A function $\phi(t_1, t_2)$ defined in $-\infty < t_1, t_2 < \infty$ is a frequency function, if the function

$$\alpha(t_1, t_2) = \int_{-\infty}^{t_1} \int_{-\infty}^{t_2} \phi(u, v) \, du \, dv \quad -\infty < t_1, t_2 < \infty$$

is a distribution function.

Any non-negative $\phi(u, v)$ for which

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(u, v) \, du \, dv = 1$$

is a frequency function.

8. The *mean* m_ϕ of a frequency function $\phi(t_1, t_2)$ is the integral

$$m_\phi = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} t_1\, t_2\, \phi(t_1, t_2)\, dt_1\, dt_2$$

9. The *partial means* of frequency function $\phi(t_1, t_2)$ are the integrals

$$m_{t_1}^{\phi}(t_2) = \int_{-\infty}^{\infty} t_1\, \phi(t_1, t_2)\, dt_1$$

$$m_{t_2}^{\phi}(t_1) = \int_{-\infty}^{\infty} t_2\, \phi(t_1, t_2)\, dt_2$$

10. The *Variance* V_ϕ of a frequency function $\phi(t_1, t_2)$ is the integral

$$V_\phi = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(t_1 - m_{t_1}^{\phi}(t_2)\right)^2 \left(t_2 - m_{t_2}^{\phi}(t_1)\right)^2 \phi(t_1, t_2)\, dt,\, dt_2$$

11. The *Characteristic Function* $\chi_\alpha(s_1, s_2)$ of a distribution function $\alpha(t_1, t_2)$ is the integral

$$\chi_\alpha(s_1, s_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\vec{s}\cdot\vec{t}}\, d_{t_1}\, \alpha(t_1, t_2)\, d_{t_2}\, \alpha(t_1, t_2)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\vec{s}\cdot\vec{t}}\, \frac{\partial \alpha}{\partial t_1} \frac{\partial \alpha}{\partial t_2}\, dt_1\, dt_2$$

12. The *Characteristic function* $\chi_\phi(s_1, s_2)$ of a frequency function $\phi(t_1, t_2)$ is the integral

$$\chi_\phi(s_1, s_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\vec{s}\cdot\vec{t}}\, \phi(t_1, t_2)\, dt_1\, dt_2$$

13. We now refer to (1·1) as

(13·1) $$f(x) = G^*(\phi(x)$$

and represent (2·1) as

(13·2) $$f(x) = G \# \phi(x)$$

With these notations $*$, $\#$, we can represent our convolutions, in case of two variables,

(13·3) $$f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(x - u, y - v)\, \phi(u, v)\, du\, dv$$

(13·4) $$f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(x - u, y - v)\, d_u\phi(u, v)\, d_v\phi(u, v)$$

as

(13·5) $$f(x, y) = G * \phi(x, y)$$

(13·6) $$f(x, y) = G \# \phi(x, y)$$

respectively.

14. *Theorem* :

If $\alpha(t_1, t_2)$, $\beta(t_1, t_2)$ are distribution functions with partial means $m^{\alpha}{}_{t_1}(t_2)$, $m^{\alpha}{}_{t_2}(t_1)$ and $m^{\beta}{}_{t_1}(t_2)$, $m^{\beta}{}_{t_2}(t_1)$, having variance V_α and V_β respectively, then $\alpha \# \beta(t_1, t_2)$ has the partial means and variance $m^{\alpha}{}_{t_1}(t_2) + m^{\beta}{}_{t_1}(t_2)$; $m^{\alpha}{}_{t_2}(t_1) + m^{\beta}{}_{t_2}(t_1)$ and $V_\alpha + V_\beta$ respectively.

This argument required for proving this theorem follows the pattern of a similar theorem in case of single variable Stieltjes Convolution, referred to by Hirschman and Widder, [3 ; p. 23] and established by Widder, [4, (1946) ; 257].

15. *Theorem* :

If the integrals

$$f(s_1, s_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\vec{s}\cdot\vec{t}}\, d_{t_1}\, \alpha(t_1, t_2)\, d_{t_2}\, \alpha(t_1, t_2)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\vec{s}\cdot\vec{t}}\, \frac{\partial \alpha}{\partial t_1}\, \frac{\partial \alpha}{\partial t_2}\, dt_1\, dt_2$$

$$g(s_1, s_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\vec{s}\cdot\vec{t}} d_{t_1} f(t_1, t_2)\, d_{t_2}\, \beta(t_1, t_2)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\vec{s}\cdot\vec{t}}\, \frac{\partial \beta}{\partial t_1}\, \frac{\partial \beta}{\partial t_2}\, dt_1\, dt_2$$

converge absolutely for a common $\vec{s} = s_1 \hat{i} + s_2 \hat{j}$, then for that value

$$f(s_1, s_2)\, g(s_1, s_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\vec{s}\cdot\vec{t}}\, dt_1\, \gamma(t_1, t_2)\, d_{t_2}\, \gamma(t_1, t_2)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\vec{s}\cdot\vec{t}}\, \frac{\partial \gamma}{\partial t_1}\, \frac{\partial \gamma}{\partial t_2}\, dt_1\, dt_2$$

where $\gamma(t_1, t_2) = \alpha(t_1, t_2) \# \beta(t_1, t_2)$.

The proof can be developed on the lines of Hirschman and Widder, [3 ; p. 23], who have taken up the functions of a single variable for their discussion.

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References

1. Ambasht, J. P. Generalization of Dirac's δ-function and Green's G-function. *Paper No.* A 1210/6-5-67/*Nat. Acad. Sc., India.*
2. Jain, M. K. Convolution Transform I. *Unpublished.*
3. Hirschman, I. I. and Widder, D. V. The Convolution Transform. *Princeton Univ. Press,* 1955.
4. Widder, D. V. The Laplace Transform. 2*nd ed., Princeton,* 1946.

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A Class of Integral Equations

By

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Abstract

The solutions of the integral equations involving generalized Laguerre polynomial and Jacobi polynomial are obtained by the application of the Rodrigues formulae.

1. The integral equation

$$\int_1^x P_n\left(\frac{x}{t}\right) g(t)\, dt = f(x),\ 1 < x < x_0.$$

Where $P_n(x/t)$ is the Legendre polynomial of order n was solved by A. Erdélyi[1], who applied Rodrigues' formula for the Legendre polynomial. Shrivastava[2] has reduced the solution of certain dual integral equations to the problem of solving an integral equation involving Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ in a special case when $\alpha = 0$, which he has solved by applying Rodrigues' formula for the Jacobi polynomial.

The object of the present paper is to give the solutions of the integral equations involving generalized Laguerre polynomial, and Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ in the most general case, respectively, by the application of the Rodrigues' formulae.

2. First we consider the integral equation

$$\int_1^x (x-y)^\alpha L_n^\alpha (x-y)\, g(y)\, dy = f(x),\ 1 < x < x_0. \tag{2·1}$$

Where $L_n^\alpha (x-y)$ is the Laguerre polynomial of order n. We give the solution in the case when α is some non-negative integer.

Since $L_0^\alpha (x-y) = 1$, we easily obtain

$$f' = g, \text{ when } \alpha = 0,\ n = 0 \text{ and} \tag{2·2}$$

$$g(x) = \frac{1}{(\alpha)!} f^{\alpha+1}(x), \text{ when } \alpha \neq 0,\ n = 0. \tag{2·3}$$

Let us assume when $n \geqslant 1$,

$f(x)$ is absolutely continuous in the interval $1 < x < x_0$, (2·4)

$$f(1) = 0. \tag{2·5}$$

The conditions (2·4) and (2·5) are necessary for the unique solution of the integral equation (2·1).

Multiplying both the sides of the equation (2·1) by

e^{-x}, we get

$$\int_1^x e^{-x}\,(x-y)^{\alpha}\,L_n^{\alpha}\,(x-y)\,g(y)\,dy = f(x)\,e^{-x}. \tag{2·6}$$

We obtain the following relation from Rodrigues' formula for the generalized Laguerre polynomial.

$$e^{-x}\,(x-y)^{\alpha}\,L_n^{\alpha}\,(x-y) = \frac{1}{n!}\cdot\left(\frac{d}{dx}\right)^n\{e^{-x}\,(x-y)^{n+\alpha}\} \tag{2·7}$$

From (2·6) and (2·7), we have

$$\left(\frac{d}{dx}\right)^n \int_1^x e^{-x}\,(x-y)^{n+\alpha}\,g(y)\,dy = n!\,.\,f(x)\,e^{-x}$$

and by n repeated integrations, it follows that

$$\int_1^x e^{-x}\,(x-y)^{n+\alpha}\,g(y)\,dy = n\int_1^x (x-u)^{n-1}\,e^{-u}\,f(u)\,du \tag{2·8}$$

Or

$$\int_1^x (x-y)^{n+\alpha}\,g(y)\,dy = n\int_1^x (x-u)^{n-1}\,e^{x-u}\,f(u)\,du$$

Differentiating the above equation $(n+\alpha+1)$ times with respect to x anp dividing by $(n+\alpha)$!, we get

$$g(x) = \frac{n}{(n+\alpha)!}\cdot\left(\frac{d}{dx}\right)^{n+\alpha+1}\int_1^x (x-u)^{n-1}\,e^{x-u}\,f(u)\,du. \tag{2·9}$$

When $\alpha = 0$, it is the case of Simple Laguerre polynomial. The relation (2·9) gives the solution of the integral equation (2·1), under the conditions (2·4) and (2·5).

3. Now we consider the integral equation

$$\int_1^x (y-x)^{\alpha}\,y^n\,P_n^{(\alpha,\beta)}\left(\frac{2x}{y}-1\right)g(y)\,dy = f(x),\ 1 < x < x_0 \tag{3·1}$$

where $P_n^{(\alpha,\beta)}\left(\frac{2x}{y}-1\right)$ is the Jacobi polynomial of order n. Here also, the solution is being given in the case when α is some non-negative integer.

Since $P_0^{(\alpha,\beta)}\left(\frac{2x}{y}-1\right) = 1$, we easily obtain

$$f'(y) = g(y), \text{ when } \alpha = 0,\ n = 0 \text{ and} \tag{3·2}$$

$$g(x) = \frac{f^{\alpha+1}(x)}{(\alpha)!} \text{ when } \alpha \neq 0,\ n = 0. \tag{3·3}$$

Let us assume when $n \geqslant 1$, the conditions (2·4) and (2·5) are satisfied because these conditions are necessary for the unique solution of the integral equation (3·1).

Multiplying both the sides of the equation (3·1) by x^{β}, we get

$$(3\cdot4) \qquad \int_1^x (y-x)^{\alpha} y^n x^{\beta} P_n^{(\alpha,\beta)}\left(\frac{2x}{y} - 1\right) g(y)\, dy = x^{\beta} f(x)$$

We obtain the following relation from Rodrigues' formula for the Jacobi polynomial,

$$(3\cdot5) \qquad (y-x)^{\alpha} x^{\beta} y^n P_n^{(\alpha,\beta)}\left(\frac{2x}{y} - 1\right) = \frac{(-1)^n}{n!} \left(\frac{d}{dx}\right)^n \{(y-x)^{n+\alpha}.\, x^{n+\beta}\}$$

From (3·4) and (3·5) we have

$$\left(\frac{d}{dx}\right)^n \int_1^x (y-x)^{n+\alpha} x^{n+\beta} g(y)\, dy = (-1)^n\, n!\, x^{\beta} f(x)$$

and by n repeated integrations, it follows that

$$(3\cdot6) \qquad \int_1^x (y-x)^{n+\alpha} x^{n+\beta} g(y)\, dy = (-1)^{n+\alpha}\, n \int_1^x (x-u)^{n-1} u^{\beta} f(u)\, du$$

$$\text{Or} \quad \int_1^x (y-x)^{n+\alpha} g(y)\, dy = (-1)^{n+\alpha}\, n \int_1^x x^{-(n+\beta)} (x-u)^{n-1} u^{\beta} f(u)\, du.$$

Differentiating the above equation $(n+\alpha+1)$ times with respect to x and dividing by $(n+\alpha)!$, we get

$$(3\cdot7) \qquad g(x) = \frac{n}{(n+\alpha)!} \cdot \left(\frac{d}{dx}\right)^{n+\alpha+1} \int_1^x x^{(n+\beta)} (x-u)^{n-1} u^{\beta} f(u)\, du.$$

It has been already mentioned that the case $\alpha = 0$ was discussed by[2]. This case is the special case of what has been shown above. The relation (3·7) gives the solution of the integral equation (3·1), under the conditions (2·4) and (2·5).

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References

1. Erdélyi, A. An integral equation involving Legendre's polynomial. *Amer. Math. Monthly*, **70**(6) June-July, 1963.
2. Shrivastava, K. N. On certain dual integral equations. *The Journal of the Indian Mathematical Society, New-series*, **29**(1-2), Jan-June, 1965.
3. Rainville, E. D. *Special function*, 1960.
4. Sneddon, I. N. *Special functions of Mathematical Physics and Chemistry*, 1961.

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Adsorption of Iodide by Soils

By

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Abstract

Adsorption of iodide from potassium iodide by two samples of soils was studied in aqueous medium at 28°C. It was observed that among the soil constituents carbon, iron and aluminium are the important factors responsible for iodide adsorption by soils.

It is generally known that iodine, phosphorus and copper are not present in soils in proper amounts and generally lack in these systems. Except for root crops, iodine, however, is not an essential element for plants and normally grow without any addition of it to the soil. Notwithstanding, iodine manuring is essential where it is actually deficient since the element is required by animals and Homo sapiens for the proper functioning of their thyroid glands. The gland is known to produce an internal secretion of a non amorphous hormone, thyroxine, containing about 65% of iodine. Inadquate supply of this element causes enlargement of the thyroid gland (simple or endemic goiter) as well as other disorders of the body (exophthalmic goiter). In some way or the other, animals and human beings get their supply of iodine through plants and it is therefore, essential that the iodine status of the soil be maintained to the optimum.

Iodine is commonly added to soils in the form of potassium iodide, potassium iodate, raw phosphate rock, Chile nitrate and limestone especially dolomites. Normally, potassium iodide is preferred to others. From indirect evidence, it appears that soon after its addition to soils, it is rapidly converted to forms unavailable to plants.

Rozen (1951) studied iodine and bromine adsorption on mineral adsorbents from aqueous solutions and found increased absorption of these elements in presence of chlorides and sulphates of sodium, potassium, magnesium and calcium. Iodine adsorption by aluminium hydroxide was comparatively larger than bromine. De (1961) studied the adsorptive behaviour of iodide with a few silicate minerals and determined the primary factors responsible for the locking effect on iodide in presence of these minerals. The study of iodide adsorption by soils, however, has not yet been done in any great detail.

In the present study the nature of iodide adsorption by two U. P. soils has been investigated and the more important soil components responsible for its locking effect have been determined.

Experimental

The soil samples A and B were collected from the Allahabad University campus. Soil A was under vegetation, infested mostly with *Achyranthus aspera* and *Cyanodon dactylon*. Soil B was without vegetation and for a greater part of the day was exposed to the direct rays of the sun. Chemical analysis of the soils are given below.

Percent chemical analysis of Soil A and Soil B.

		Soil A	Soil B
Total C		1·035	0·225
Total sesquioxide		10·834	10·234
Fe_2O_3		4·245	5·223
Total Ca		1·48	0·41
Total P_2O_5		0·5778	0·194
Ex. (m.e/100 g)	Ca	15·7	24·0
	Mg	3·82	5·27

1·0 gm. of the soil samples (air-dried and 100-mesh sieved) were taken in different 100 ml. volumetric flasks and to these were added required quantities of the iodide solutions. The volume was made up by adding carbon dioxide-free distilled water and after thorough shaking, the flasks were kept for 24 hours in a thermostat maintained at 28°C. Next day, by taking suitable aliquot portions, the supernatant liquid was analysed for the equilibrated iodide contents using deci-normal silver nitrate solution and di-iododimethyl flourescein as an indicator (Vogel, 1951). To see the effects of the complexing agents, in all cases these were added to the soils and kept for 24 hours in a thermostat at 28°C ; then the iodide solutions were added and after the volume was made up were similarly kept in a thermostat. 10 ml. 2% aqueous solution of cupferron and 10 ml 2% acetic acid solution of 8-hydroxyquinoline were employed as complexing agents. The soils were leached with normal sodium and ammonium chloride solutions in order to see to a greater part the effect of exchangeable divalents. 60% alcohol was used to clear off the excess of sodium and ammonium chloride solutions from the soil samples. The leached soil samples were carefully dried in an oven at a low temperature (Ca. 60°C) and then used for the adsorption experiments. To see the effect of organic matter, the soils were similarly used after they were ignited for different times.

The results reported here are in terms of milligrams of iodide per 100 ml. solution.

Results and Discussion

Soil B recorded higher adsorption of iodide than soil A. In presence of the complexing agents, iodide adsorption decreased more in Soil B than in A. Soils leached with ammonium chloride solution recorded not as much decrease in iodide adsorption as those leached with sodium chloride. Also, this decrease was less in soil A than in soil B Soils ignited for 30 minutes recorded slight increase in iodide adsorption but as the time of ignition was raised, this increase in adsorption started diminishing perceptibly.

The figure (Freundlich's plot, not shown to economise space) showed that with soil B, the iodide adsorption was not unimolecular, and that from 0·0255M there was distinct formation of additional layers. Adsorption of iodide by soil A, however, appears to be unimolecular. It will be of interest to mention here that Reyerson and Cameron (1936) obtained monomolecular layer adsorption of iodine and bromine after 24 hours on activated carbon Since soil A is about 5 times richer in carbon than soil B, it very clearly points to the fact that as the carbon contents increase, iodide adsorption on such systems tends to be unimolecular and the adsorption isotherms approach very near to a straight line. Also the observations show (Table I) that with the increase of carbonaceous materials in soils the extent of iodide adsorption by such systems decrease.

TABLE 1

Adsorption of iodide from potassium iodide by soils at 28°C

Initial conc. (molar)	Adsorption (x/m) (mg./100 ml. solution)	
	Soil A	Soil B
0·0306	1005·04	2215·29
0·0255	776·63	1050·73
0·0204	513·69	1327·89
0·0153	456 84	963·17
0 0102	96·95	708·10
0·0051	137·05	286·79

TABLE 2

Adsorption of iodide from potassium iodide by soils in presence of complexing agents at 28°C

Initial conc. (molar)	Adsorption (x/m) (mg./100 ml. solution)			
	Soil A		Soil B	
	Cupferron	8-hydroxyquinoline	Cupferron	8-hydroxyquinoline
0 0306	825·28	575·32	1616·26	846·38
0·0153	347·21	234·84	704·83	317·74

TABLE 3

Adsorption of iodide from potassium iodide by soils as affected by the removal of their exchangeable divalents

Initial conc. (molar)	Adsorption (x/m) (mg./100 ml. solution)			
	Soil A		Soil B	
	NaCl	NH_4Cl	NaCl	NH_4Cl
0·0306	580·41	680·63	864·26	1082·56
0·0153	258·31	301·24	306·24	446·82

TABLE 4

Adsorption of iodide from potassium iodide by ignited soils at 28°C

Initial conc. (molar)	Adsorption (x/m) (mg./100 ml. solution)					
	Soil A			Soil B		
	1/2 hour	1 hour	1½ hours	1/2 hour	1 hour	1½ hours
0·0306	1421·23	1008·25	921·81	2765·34	2180·11	1981·25
0·0153	651·21	446·13	365·34	1200·13	945·24	830·14

The role of iron and aluminium in the phenomenon of iodide adsorption may be observed in table 2. The iodide adsorption by the two soil samples diminished appreciably as they were complexed with cupferron and 8–hydroxyquinoline. Cupferron was used to complex iron and 8-hdroxyquinoline both for iron and aluminium to determine the role of the latter. With both the soils, in presence of 8-hydroxyquinoline, the reduction in adsorption was far greater in amount than what was observed with cupferron. It is therefore, aluminium with its various compounds in soil that is camparatively more important than the others in the dynamics of iodide removal from the liquid phase. The interesting fact is that the adsorption by soil B is reduced to a greater extent than soil A. It appears that most of the iron and aluminium in soil A was already complexed with the organic matter and this was not in soil B which contained about 5 times lesser carbon. This is also the reason why soil B recorded higher adsorption than soil A. In table 3, greater reduction in the adsorption of iodide by soil B than soil A when leached with ammonium and sodium chloride solutions is undoubtedly due to more uncomplexed form of aluminium, iron, calcium and magnesium in the former than the latter. There is always appreciable release of these uncomplexed tri- and divalents from the soil when treated with sodium and ammonium chloride solutions.

References

1. Rozen, B. Ya. *Doklady Akad. Nauk S. S. R.*, **81**, 243, 1951.
2. Reyerson, L. H. and Cameron, A. E. *J. Phys. Chem.*, **40**(2), 233, 1936.
3. De, S. K. *Proc. nat. Acad. of Sci. India*, **30A**, 195, 1960.

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On some integrals involving generalized Legendre's associated functions and H-functions

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Abstract

The object of this paper is to evaluate some integrals involving product of the generalized Legendre's associated functions and the H-function. The generalized Legendre's associated functions reduce to associated Legendre functions on setting $m = n$ and to Legendre functions on setting $m = n = 0$. Also on specializing the parameters of the H-function which is very general function, we get various other functions. Thus on specializing the parameters of these functions in the integrals, we get many interesting new results.

1. Introduction. In[7], Meulenbeld and Kuipers defined generalized Legendre's associated functions $P_k^{m,n}(z)$ and $Q_k^{m,n}(z)$ as linearly two independent solutions of the differential equation

$$(1\cdot1)\qquad (1-z^2)\frac{d^2\omega}{dz^2} - 2z\frac{d\omega}{dz} + \left\{k(k+1) - \frac{m^2}{2(1-z)} - \frac{n^2}{2(1+z)}\right\}\omega = 0$$

specified either in the form of integrals over double circuits, or in terms of hypergeometric functions valid in restricted areas of the z-plane in which a cross-cut exists along the real-axis from 1 to ∞.

Further, they have restricted $z = x$ to lie in the interval $-1 < x < 1$ and defined $P_k^{m,n}(x)$ for unrestricted values of k, m and n except if m is a positive integer [10, p. 560(3)] as

$$(1\cdot2)\qquad P_k^{m,n}(x) = \frac{(1+x)^{\frac{n}{2}}}{(1-x)^{\frac{m}{2}}\Gamma(1-m)}\,{}_2F_1\left[\begin{matrix} k-\frac{m-n}{2}+1, -k-\frac{m-n}{2}; \\ 1-m; \end{matrix}\ \frac{1-x}{2}\right].$$

If m is a non-negative integer and $-k+\frac{m+n}{2}$ and $-k-\frac{m-n}{2}$ are not integers [9, p. 148(3)], then

$$(1\cdot3)\qquad P_k^{m,n}(x) = \frac{(1+x)^{\frac{n}{2}}(1-x)^{\frac{m}{2}}\,\Gamma\left(k+\frac{m+n}{2}+1\right)\Gamma\left(-k+\frac{m+n}{2}\right)}{2^m\,\Gamma(1+m)\,\Gamma\left(k-\frac{m-n}{2}+1\right)\Gamma\left(-k-\frac{m-n}{2}\right)}$$

$$\times {}_2F_1\left[\begin{matrix} k+\frac{m+n}{2}+1, -k+\frac{m+n}{2} \\ m+1 \end{matrix}; \frac{1-x}{2}\right].$$

If $k \pm \frac{m \pm n}{2}$ are positive integers and m is a non-negative integer, then [8, p. 25(6)]

$$(1\cdot4) \qquad P_k^{m,n}(x) = \frac{(1+x)^{\frac{n}{2}}(1-x)^{\frac{m}{2}}\,\Gamma\left(k+\frac{m+n}{2}+1\right)\Gamma\left(k+\frac{m-n}{2}+1\right)}{2^m\,\Gamma(m+1)\,\Gamma\left(k-\frac{m-n}{2}+1\right)\Gamma\left(k-\frac{m+n}{2}+1\right)}$$

$$\times {}_2F_1\left[\begin{matrix} k+\frac{m+n}{2}+1, -k+\frac{m+n}{2} \\ m+1 \end{matrix}; \frac{1-x}{2}\right].$$

Fox [5, p. 408], introduced the H-function in the form of Mellin-Barnes type integral, which has been symbolically denoted by Gupta and Jain[6].

$$(1\cdot5) \quad H_{p,q}^{m,n}\left[x \,\middle|\, \begin{matrix} \{(a_p, \alpha_p)\} \\ \{(b_q, \beta_q)\} \end{matrix}\right] = \frac{1}{2\pi i}\int_T \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j s)}\,x^s\,ds,$$

where $\{(f_r, \gamma_r)\}$ stands for the set of the parameters $(f_1, \gamma_1), \ldots, (f_r, \gamma_r)$; x is not equal to zero and the empty product is interpreted as unity; p, q, m and n are integers satisfying $1 \leqslant m \leqslant q$, $0 \leqslant n \leqslant p$; $\alpha_j (j = 1, 2, \ldots, p)$; $\beta_j (j = 1, 2, \ldots, q)$ are positive numbers and $a_j (j = 1, 2, \ldots, p)$; $b_j (j = 1, 2, \ldots, q)$ are complex numbers such that no pole of $\Gamma(b_h - \beta_h s)$ $(h = 1, 2, \ldots, m)$ coincides with any pole of $\Gamma(1 - a_i + \alpha_i s)$ $(i = 1, 2, \ldots, n)$ *i.e.*

$$\alpha_i(b_h+\nu) \neq \beta_h(a_i - \eta - 1)$$

$$(\nu, \eta = 0, 1, \ldots; h = 1, 2, \ldots, m; i = 1, 2, \ldots, n).$$

According to Braaksma [3, p. 240], H-function makes sense and defines an analytic function of x when

$$\sum_1^q (\beta_j) - \sum_1^p (\alpha_j) \equiv \mu > 0, \; x \neq 0; \text{ if } \mu = 0, \; 0 < |x| < \beta^{-1} \text{ where } \beta = \prod_1^p (\alpha_j)^{\alpha_j} \prod_1^q (\beta_j)^{-\beta_j}.$$

From [3, p. 279(6·5)], we have

$$(1\cdot6) \qquad H_{p,q}^{m,n}\left[x \,\middle|\, \begin{matrix} \{(a_p, \alpha_p)\} \\ \{(b_q, \beta_q)\} \end{matrix}\right] = 0\,(|x|^{\alpha}) \quad \text{for small } x,$$

where $\sum_1^q (\beta_j) - \sum_1^p (\alpha_j) \geqslant 0$ and $\alpha = Re\left(\frac{b_h}{\beta_h}\right)$ $(h = 1, 2, \ldots, m)$.

From [3, p. 246 (2·16)], we get

$$(1\cdot7) \qquad H^{m,\,n}_{p,\,q}\left[x \,\middle|\, \begin{matrix}\{(a_p, \alpha_p)\} \\ \{(b_q, \beta_q)\}\end{matrix}\right] = 0\,(|x|^{\beta}) \text{ for large } x,$$

where $\sum_1^q (\beta_j) - \sum_1^p (\alpha_j) > 0$, $\sum_1^n (\alpha_j) - \sum_{n+1}^p (\alpha_j) + \sum_1^m (\beta_j) - \sum_{m+1}^q (\beta_j) \equiv \lambda > 0$,

$|\arg x| < \frac{1}{2}\lambda\pi$ and $\beta = Re\left(\frac{a_i - 1}{\alpha_i}\right)$ $(i = 1, 2, \ldots, n)$.

2. The following integrals will be established in this section

$$(2\cdot1) \qquad \int_0^1 x^{\lambda-1}(1+x)^{-\frac{n}{2}}(1-x)^{\sigma} P^{m,\,n}_{k}(x)\, H^{l,\,u}_{r,\,s}\left[zx^{\delta}(1-x)^{\mu} \,\middle|\, \begin{matrix}\{(a_r, \alpha_r)\} \\ \{(b_s, \beta_s)\}\end{matrix}\right] dx$$

$$= \frac{1}{2^m\,\Gamma\left(k-\frac{m-n}{2}+1\right)\Gamma\left(-k-\frac{m-n}{2}\right)} \sum_{N=0}^{\infty} \frac{\Gamma\left(-k+\frac{m+n}{2}+N\right)\Gamma\left(1+k+N+\frac{m+n}{2}\right)}{2^N\,\Gamma(N+1)\,\Gamma(1+m+N)}$$

$$\times\, H^{l,\,u+2}_{r+2,\,s+1}\left[z \,\middle|\, \begin{matrix}(1-\lambda,\,\delta),\left(-\sigma-N-\frac{m}{2},\,\mu\right),\{(a_r, \alpha_r)\} \\ \{(b_s, \beta_s)\},\left(-\lambda-\sigma-N-\frac{m}{2},\,\delta+\mu\right)\end{matrix}\right],$$

provided $Re(\lambda) > 0$, m is a non-negative integer, $-k+\frac{m+n}{2}$ and $-k-\frac{m-n}{2}$ are not integers, $Re(\lambda+\delta b_h/\beta_h) > 0$ $(h = 1, 2, \ldots, l)$,

$$\sum_1^s (\beta_j) - \sum_1^r (\alpha_j) \geqslant 0,\ \sum_1^l (\beta_j) + \sum_{l+1}^s (\beta_j) + \sum_1^u (\alpha_j) - \sum_{u+1}^r (\alpha_j) \equiv \phi > 0,\ |\arg z| < \tfrac{1}{2}\phi\pi,$$

and $\delta > 0$, $\mu \geqslant 0$ (or $\delta \geqslant 0$, $\mu > 0$).

$$(2\cdot2) \qquad \int_0^1 x^{\lambda-1}(1+x)^{-\frac{n}{2}}(1-x)^{\sigma+\frac{m}{2}} P^{m,\,n}_{k}(x)\, H^{l,\,u}_{r,\,s}\left[zx^{\delta}(1-x)^{\mu} \,\middle|\, \begin{matrix}\{(a_r, \alpha_r)\} \\ \{(b_s, \beta_s)\}\end{matrix}\right] dx$$

$$= \sum_{N=0}^{\infty} \frac{\left(k-\frac{m-n}{2}+1\right)_N \left(-k-\frac{m-n}{2}\right)_N}{2^N\,\Gamma(N+1)\,\Gamma(1-m+N)}\, H^{l,\,u+2}_{r+2,\,s+1}\left[z \,\middle|\, \begin{matrix}(1-\lambda,\delta),(-\sigma-N,\mu),\{(a_r,\alpha_r)\} \\ \{(b_s,\beta_s)\},(-\lambda-\sigma-N,\,\delta+\mu)\end{matrix}\right],$$

where m is not a positive integer, $Re(\lambda) > 0$, $Re(\lambda+\delta\frac{b_h}{\beta_h}) > 0$ $(h = 1, 2, \ldots, l)$.

$\sum_{1}^{s}(\beta_j)-\sum_{1}^{r}(\alpha_j)\geqslant 0,\ \sum_{1}^{l}(\beta_j)-\sum_{l+1}^{s}(\beta_j)+\sum_{1}^{u}(\alpha_j)-\sum_{u+1}^{r}(\alpha_j)\equiv\phi>0,\ |\arg z|<\frac{1}{2}\phi\pi$

and $\delta>0,\ \mu\geqslant 0$ (or $\delta\geqslant 0,\ \mu>0$).

$$(2\cdot3)\qquad \int_0^1 x^{\lambda-1}(1+x)^{-\frac{n}{2}}(1-x)^{\sigma}P_k^{m,n}(x)\,H^{l,u}_{r,s}\left[\frac{zx\delta}{(1-x)\delta}\,\middle|\,\begin{matrix}\{(a_r,\alpha_r)\}\\ \{(b_s,\beta_s)\}\end{matrix}\right]dx$$

$$=\frac{1}{2^m\,\Gamma\left(k-\frac{m-n}{2}+1\right)\Gamma\left(-k-\frac{m-n}{2}\right)}\sum_{N=0}^{\infty}\frac{\Gamma\left(-k+\frac{m+n}{2}+N\right)\Gamma\left(1+k+N+\frac{m+n}{2}\right)}{2^N\,\Gamma(N+1)\,\Gamma(m+1+N)\quad\Gamma\left(1+\lambda+\sigma+N+\frac{m}{2}\right)}$$

$$\times H^{l+1,\,u+1}_{r+1,\,s+1}\left[z\,\middle|\,\begin{matrix}(1-\lambda,\delta),\{(a_r,\alpha_r)\}\\ (1+\sigma+N+\frac{m}{2},\delta),\{(b_s,\beta_s)\}\end{matrix}\right]$$

provided m is a non-negative integer, $-k+\frac{m+n}{2}$ and $-k-\frac{m-n}{2}$ are not integers, $\delta>0$, $Re(\lambda)>0$, $Re(\lambda+\delta b_h/\beta_h)>0$ $(h=1,2,\ldots\ldots,l)$

$\sum_{1}^{s}(\beta_j)-\sum_{1}^{r}(\alpha_j)\geqslant 0,\ \sum_{1}^{l}(\beta_j)-\sum_{l+1}^{s}(\beta_j)+\sum_{1}^{u}(\alpha_j)-\sum_{u+1}^{r}(\alpha_j)\equiv\phi>0,\ |\arg z|<\frac{1}{2}\phi\pi$

$$(2\cdot4)\qquad \int_0^1 x^{\lambda-1}(1+x)^{-\frac{n}{2}}(1-x)^{\sigma+\frac{m}{2}}P_k^{m,n}(x)\,H^{l,u}_{r,s}\left[\frac{zx\delta}{(1-x)\delta}\,\middle|\,\begin{matrix}\{(a_r,\alpha_r)\}\\ \{(b_s,\beta_s)\}\end{matrix}\right]dx$$

$$=\sum_{N=0}^{\infty}\frac{\left(k-\frac{m-n}{2}+1\right)_N\left(-k-\frac{m-n}{2}\right)_N}{2^N\,\Gamma(N+1)\,\Gamma(1-m+N)\,\Gamma(1+\lambda+\sigma+N)}H^{l+1,\,u+1}_{r+1,\,s+1}\left[z\,\middle|\,\begin{matrix}(1-\lambda,\delta)\{(a_r,\alpha_r)\}\\ (1+\sigma+N,\delta),\{(b_s,\beta_s)\}\end{matrix}\right]$$

where m is not a positive integer, $\delta>0$, $Re(\lambda)>0$, $Re(\lambda+\delta b_h/\beta_h)>0$

$(h=1,2,\ldots\ldots,l)$, $\sum_{1}^{s}(\beta_j)-\sum_{1}^{r}(\alpha_j)\geqslant 0$, $\sum_{1}^{l}(\beta_j)-\sum_{l+1}^{s}(\beta_j)+\sum_{1}^{u}(\alpha_j)-\sum_{u+1}^{r}(\alpha_j)\equiv\phi>0$, $|\arg z|<\frac{1}{2}\phi\pi$.

$$(2\cdot5)\qquad \int_0^1 t^{\rho-\frac{m}{2}-1}(1-t)^{\frac{n}{2}}P_k^{m,n}(1-2t)\,H^{l,u}_{r,s}\left[zt^{\delta}\,\middle|\,\begin{matrix}\{(a_r,\alpha_r)\}\\ \{(b_s,\beta_s)\}\end{matrix}\right]dt$$

$$=\frac{2^{\frac{n-m}{2}}\,\Gamma\left(1+k+\frac{m+n}{2}\right)}{\Gamma\left(k-\frac{m+n}{2}+1\right)}$$

$$\times H^{l+1,\,u+1}_{r+2,\,s+2}\left[z\,\middle|\,\begin{matrix}(1-\rho,\delta),\{(a_r,\alpha_r)\},(1+m-\rho,\delta)\\ (1+k-\rho+\frac{m-n}{2},\delta),\{(b_s,\beta_s)\},(-k-\rho+\frac{m-n}{2},\delta)\end{matrix}\right]$$

where m, $k \pm \frac{m \pm n}{2}$ are positive integers, $\delta > 0$, $Re(n) > -1$, $Re(\rho) > 0$,

$Re(\rho + \delta b_h/\beta_h) > 0$ $(h = 1, 2, \ldots\ldots, l)$, $\sum_{1}^{s} (\beta_j) - \sum_{1}^{r} (\alpha_j) \geqslant 0$, $\sum_{1}^{l} (\beta_j) -$

$$\sum_{l+1}^{s} (\beta_j) + \sum_{1}^{u} (\alpha_j) - \sum_{u+1}^{r} (\alpha_j) \equiv \phi > 0, \quad |\arg z| < \tfrac{1}{2}\phi\pi.$$

$$\int_0^1 t^{\rho+\frac{m}{2}-1} (1-t)^{\frac{n}{2}} P_k^{m,n}(1-2t)\, H_{r,s}^{l,u}\left[zt^{\delta} \,\middle|\, \begin{matrix} \{(a_r, \alpha_r)\} \\ \{(b_s, \beta_s)\} \end{matrix}\right] dt \tag{2·6}$$

$$= \frac{2^{\frac{n-m}{2}}\, \Gamma\left(1+k+\frac{m+n}{2}\right)}{\Gamma\left(1+k-\frac{m+n}{2}\right)}$$

$$\times H_{r+2,\, s+2}^{l+1,\, u+1}\left[z \,\middle|\, \begin{matrix} (1-\rho,\ \delta),\ \{(a_r, \alpha_r)\},\ (1-m-\rho,\ \delta) \\ \left(1+k-\rho-\frac{m+n}{2},\ \delta\right),\ \{(b_s, \beta_s)\},\ \left(-k-\rho-\frac{m+n}{2},\ \delta\right) \end{matrix}\right]$$

provided m is not a positive integer, $\delta > 0$, $Re(n) > -1$, $Re(\rho) > 0$,

$Re(\rho + \delta b_h/\beta_h) > 0$ $(h = 1, 2, \ldots\ldots, l)$,

$$\sum_{1}^{s} (\beta_j) - \sum_{1}^{u} (\alpha_j) \geqslant 0, \quad \sum_{1}^{l} (\beta_j) - \sum_{l+1}^{s} (\beta_j) + \sum_{1}^{u} (\alpha_j) - \sum_{u+1}^{r} (\alpha_j) \equiv \phi > 0, \quad |\arg z| < \tfrac{1}{2}\phi\pi.$$

Proof: To evaluate the integral (2·1), expressing the H-function in Mellin-Barnes type of integral (1·5) and interchanging the order of integration, which is justifiable due to the absolute convergence of the integrals involved in the process, we get,

$$\frac{1}{2\pi i} \int_T \frac{\prod_{j=1}^{l} \Gamma(b_j - \beta_j \xi) \prod_{j=1}^{u} \Gamma(1 - a_j + \alpha_j \xi)}{\prod_{j=l+1}^{s} \Gamma(1 - b_j + \beta_j \xi) \prod_{j=u+1}^{r} \Gamma(a_j - \alpha_j \xi)}\, z^{\xi} \times \tag{2·7}$$

$$\times \int_0^1 x^{\lambda + \delta\xi - 1} (1+x)^{-\frac{n}{2}} (1-x)^{\sigma + \mu\xi}\, P_k^{m,n}(x)\, dx\, d\xi,$$

now in the inner-integral substituting the value of $P_k^{m,n}(x)$ from (1·3) and then evaluating it with the help of [11, p. 48 (6*ii*)], we have

$$\frac{\Gamma\left(k + \frac{m+n}{2} + 1\right) \Gamma\left(-k + \frac{m+n}{2}\right)}{2^m\, (m+1)\, \Gamma\left(k - \frac{m-n}{2} + 1\right) \Gamma\left(-k - \frac{m-n}{2}\right)} \tag{2·8}$$

$$\times \frac{1}{2\pi i}\int_T \frac{\prod_{j=1}^{l}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{u}\Gamma(1-a_j+\alpha_j\xi)\,\Gamma(\lambda+\delta\xi)}{\prod_{j=l+1}^{s}\Gamma(1-b_j+\beta_j\xi)\prod_{j=u+1}^{r}\Gamma(a_j-\alpha_j\xi)}\,\frac{\Gamma\left(\sigma+\frac{m}{2}+\mu\xi+1\right)}{\Gamma\left(1+\lambda+\sigma+\frac{m}{2}+\delta\xi+\mu\xi\right)}$$

$$\times {}_3F_2\left[\begin{matrix}-k+\frac{m+n}{2},\ k+\frac{m+n}{2}+1,\ 1+\sigma+\frac{m}{2}+\mu\xi\\ m+1,\ 1+\lambda+\sigma+\frac{m}{2}+\delta\xi+\mu\xi\end{matrix};\frac{1}{2}\right] z^{\xi}\,d\xi,$$

expressing the hypergeometric function as series and changing the order of integration and summation in view of [4, p. 500] ; which is permissible under the conditions given in (2·1), we obtain

$$(2\cdot9)\quad \frac{1}{2^m\,\Gamma\left(k-\frac{m-n}{2}+1\right)\Gamma\left(-k-\frac{m-n}{2}\right)}\sum_{\mathcal{N}=0}^{\infty}\frac{\Gamma\left(-k+\frac{m+n}{2}+\mathcal{N}\right)\Gamma\left(1+k+\mathcal{N}+\frac{m+n}{2}\right)}{2^{\mathcal{N}}\,\Gamma(\mathcal{N}+1)\,\Gamma(m+\mathcal{N}+1)}$$

$$\times\frac{1}{2\pi i}\int_T \frac{\prod_{j=1}^{l}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{u}\Gamma(1-a_j+\alpha_j\xi)\,\Gamma(\lambda+\delta\xi)\,\Gamma(1+\sigma+\mathcal{N}+\frac{m}{2}+\mu\xi)}{\prod_{j=l+1}^{s}\Gamma(1-b_j+\beta_j\xi)\prod_{j=u+1}^{r}\Gamma(a_j-\alpha_j\xi)\,\Gamma(1+\lambda+\sigma+\mathcal{N}+\frac{m}{2}+\delta\xi+\mu\xi)}\,z^{\xi}d\xi$$

now interpreting (2·9) in view of (1·5), the definition of the H-function, the value of the integral (2·1) is obtained.

Proceeding on the similar lines (2·3) can be evaluated. Also (2·2) and (2·4) can be established with the help of (1·2) and [11, p. 48 (6ii)] ; (2·5) can be obtained with the help of (1·4) and [2, p. 398 (20·2·2)] ; and (2·6) can be established by using (1·2) and [2, p. 398 (20·2·2)].

3. Particular cases : In this section we mention a few interesting particular cases.

(i) (2·2) with $\mu = 0$ reduces to

$$(3\cdot1)\quad \int_0^1 x^{\lambda-1}(1+x)^{-\frac{n}{2}}(1-x)^{\sigma+\frac{m}{2}}P_k^{m,n}(x)\,H_{r,s}^{l,u}\left[zx^{\delta}\,\middle|\,\begin{matrix}\{(a_r,\alpha_r)\}\\ \{(b_s,\beta_s)\}\end{matrix}\right]dx$$

$$=\sum_{\mathcal{N}=0}^{\infty}\frac{\left(k-\frac{m-n}{2}+1\right)_{\mathcal{N}}\left(-k-\frac{m-n}{2}\right)_{\mathcal{N}}\Gamma(1+\sigma+\mathcal{N})}{2^{\mathcal{N}}\,\Gamma(\mathcal{N}+1)\,\Gamma(1-m+\mathcal{N})}$$

$$\times H_{r+1,\,s+1}^{l,\,u+1}\left[z\,\middle|\,\begin{matrix}(1-\lambda,\delta),\{(a_r,\alpha_r)\}\\ \{(b_s,\beta_s)\},(-\lambda-\sigma-\mathcal{N},\delta)\end{matrix}\right]$$

where m is not a positive integer, $\delta > 0$, $Re(\lambda) > 0$, $Re(\lambda + \delta b_h/\beta_h) > 0$.

$(h=1, 2, \ldots, l)$, $\sum_1^s (\beta_j) - \sum_1^r (\alpha_j) \geqslant 0$, $\sum_1^l (\beta_j) - \sum_{l+1}^s (\beta_j) + \sum_1^u (\alpha_j) - \sum_{u+1}^r (\alpha_j) \equiv \phi > 0$,

$|\arg z| < \frac{1}{2}\phi\pi$.

(*ii*) Putting $\delta = 0$ in (2·2) we get

$$(3\cdot2)\qquad \int_0^1 x^{\lambda-1}(1+x)^{-\frac{n}{2}}(1-x)^{\sigma+\frac{m}{2}} P_k^{m,n}(x)\, H_{r,s}^{l,u}\left[z(1-x)^{\mu}\,\middle|\,\begin{matrix}\{(a_r,\alpha_r)\}\\ \{(b_s,\beta_s)\}\end{matrix}\right] dx$$

$$= \sum_{N=0}^{\infty} \frac{\left(k-\frac{m-n}{2}+1\right)_N \left(-k-\frac{m-n}{2}\right)_N \Gamma(\lambda)}{2^N \Gamma(N+1)\,\Gamma(1-m+N)}$$

$$\times\, H_{r+1,\,s+1}^{l,\,u+1}\left[z\,\middle|\,\begin{matrix}(-\sigma-N,\mu),\{(a_r,\alpha_r)\}\\ \{(b_s,\beta_s)\},(-\lambda-\sigma-N,\mu)\end{matrix}\right],$$

provided m is not a positive integer, $\mu > 0$, $\mathrm{Re}(\lambda) > 0$,

$\sum_1^s (\beta_j) - \sum_1^r (\alpha_j) \geqslant 0$, $\sum_1^l (\beta_j) - \sum_{l+1}^s (\beta_j) + \sum_1^u (\alpha_j) - \sum_{u+1}^r (\alpha_j) \equiv \phi > 0$, $|\arg z| < \frac{1}{2}\phi\pi$.

(*iii*) In (2·2), taking $\mu = 0$, $\sigma = -m$, $n = m$; on the right hand side expressing the H-function as Mellin-Barnes type of integral; interchanging the order of integration and summation; evaluating the series inside the integral with the help of [11, p. 25 (7·7)] and using (1·5), the definition of the H-function, we obtain

$$(3\cdot3)\qquad \int_0^1 x^{\lambda-1}(1-x^2)^{-\frac{m}{2}} P_k^m(x)\, H_{r,s}^{l,u}\left[zx^{\delta}\,\middle|\,\begin{matrix}\{(a_r,\alpha_r)\}\\ \{(b_s,\beta_s)\}\end{matrix}\right] dx$$

$$= 2^{m-\lambda}\sqrt{\pi}\; H_{r+1,\,s+2}^{l,\,u+1}\left[\frac{z}{2^{\delta}}\,\middle|\,\begin{matrix}(1-\lambda,\delta),\{(a_r,\alpha_r)\}\\ \{(b_s,\beta_s)\},\left(\frac{1}{2}+\frac{k}{2}-\frac{\lambda}{2}+\frac{m}{2},\frac{\delta}{2}\right),\left(-\frac{k}{2}-\frac{\lambda}{2}+\frac{m}{2},\frac{\delta}{2}\right)\end{matrix}\right]$$

where $m < 1$, $\delta > 0$, $Re(\lambda) > 0$, $Re(\lambda + \delta b_h/\beta_h) > 0$ $(h = 1, 2, \ldots, l)$

$\sum_1^s (\beta_j) - \sum_1^r (\alpha_j) \geqslant 0$, $\sum_1^l (\beta_j) - \sum_{l+1}^s (\beta_j) + \sum_1^u (\alpha_j) - \sum_{u+1}^r (\alpha_j) \equiv \phi > 0$, $|\arg z| < \frac{1}{2}\phi\pi$.

(*iv*) In (2·1), putting $\mu = 0$, $\sigma = -\frac{n}{2}$, replacing n by $-n$, using

$P_k^{m,-n}(x) = 2^{-n} P_k^{m,n}(x)$, setting $n = m$ and then proceeding as in (*iii*), we have

$$(3\cdot4)\qquad \int_0^1 x^{\lambda-1}(1-x^2)^{\frac{m}{2}} P_k^m(x)\, H_{r,s}^{l,u}\left[zx^{\delta}\,\middle|\,\begin{matrix}\{(a_r,\alpha_r)\}\\ \{(b_s,\beta_s)\}\end{matrix}\right] dx$$

$$= \frac{(-1)^m 2^{-\lambda-m}\sqrt{\pi}\,\Gamma(1+m+k)}{\Gamma(1-m+k)} H^{l,\,u+1}_{r+1,\,s+2}\left[\frac{z}{2\delta}\middle|\begin{matrix}(1-\lambda,\,\delta),\,\{(a_r,\,\alpha_r)\}\\ \{(b_s,\,\beta_s)\},\left(-\frac{\lambda}{2}-\frac{m}{2}-\frac{k}{2},\,\frac{\delta}{2}\right),\\ \left(\frac{1}{2}-\frac{\lambda}{2}-\frac{m}{2}+\frac{k}{2},\,\frac{\delta}{2}\right)\end{matrix}\right]$$

provided $Re(\lambda) > 0, m = 0, 1, 2, \ldots; \delta > 0,\ Re(\lambda+\delta\, b_h/\beta_h) > 0\ (h=1, 2, \ldots, l)$

$$\sum_1^s(\beta_j) - \sum_1^r(\alpha_j) \geqslant 0,\ \sum_1^l(\beta_j) - \sum_{l+1}^s(\beta_j) + \sum_1^u(\alpha_j) - \sum_{u+1}^r(\alpha_j) \equiv \phi > 0, |\arg z| < \tfrac{1}{2}\phi\pi.$$

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References

1. Bateman Project. Higher transcendental functions Vol. I. *Mc-Graw Hill*, 1953.
2. ——— Tables of Integral transforms Vol. II. *Mc-Graw Hill*, 1954.
3. Braaksma, B. L. J. Asymptotic expansions and analytic continuations for a class of Barnes-integrals. *Compos. Math.*, **15**, 239–341, 1963.
4. Bromwich, T. J. I'A. An introduction to the theory of infinite series. *Macmillan & Co.*, 1965.
5. Fox, C. The G and H-functions as symmetrical Fourier kernels. *Trans. Amer. Math. Soc.*, 98(3), 395–429, 1961.
6. Gupta, K. C. and Jain, U. C. On H-function II. *To appear in Proc. Nat. Acad. Sci., India.*
7. Kuipers, L. and Meulenbeld, B. On a generalization of Legendre's associated differential equation I and II. *Kon. Ned. Akad. Van. Wetensch. Amsterdam, Proc. Ser. A.*, **60**(4), 1957.
8. Kuipers, L. Generalized Legendre's associated functions—Integral theorems, recurrence formulas. *Monatshefte Fur Mathematik*, 24–31, 1958.
9. ——— Integral transforms in the theory of Jacobi polynomials and generalized Legendre associated functions (First part). *Kon. Ned. Akad. Van. Wetensch. Amsterdam, Proc. Ser. A.*, **62**(2), 148–152, 1959.
10. Meulenbeld, B. Generalized Legendre's associated functions for real values of the argument numerically less than unity. *Kon. Nederl. Akad. Van. Wetensch. Amsterdam, Proc. Ser. A.*, **61**(5), 557–563, 1958.
11. Sneddon, I. N. Special functions of Mathematical Physics and Chemistry. *Oliver and Boyd, New York*, 1961.

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A class of integral equations involving confluent hypergeometric function

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Abstract

The inversion theorems for integral equations involving confluent hypergeometric functions are obtained by making use of the known integrals.

1. In deriving a solution of certain aerodynamical problem, Ta Li[5] was led to a class of integral equations each of which has, as its kernel, a Chebyshev polynomial of first kind divided by the square root of the difference of two squares. Ta Li has obtained an exact solution of these int gral equations. The solution is given in the form of a singular integral involving Chebyshev polynomial. After the publication of his paper, several inversion integrals for integral equations involving Legendre, ulteraspherical Gegenbauer, Laguerre and Jacobi polynomials have been obtained[6,7,8,9,10].

Recently, by making use of the integral [2, (2), p. 402] Srivastava[4] has obtained the inversion integrals for integral equations involving confluent hypergeometric functions.

In the present paper, we obtain the inversion theorems for integral equations involving confluent hypergeometric functions, by making use of the integral [2 (3), (7) p. 402]. The method used is the same as that adopted by Srivastava[4].

2. Definitions of the functions :

(2·1) The series

$$ {}_1F_1(a\,;\,b\,;\,z) = \sum_{n=0}^{\infty} \frac{(a)_n\, z^n}{(b)_n\,(n)!} $$

in which $b \neq$ zero or a negative integer is convergent for all finite z. The function ${}_1F_1\,(a\,;\,b\,;\,z)$ is known as the

Pochhammer – Barnes confluent hypergeometric function.

The function $w = {}_1F_1\,(a\,;\,b\,;\,z)$ is a solution of the differential equation

$$ z\,\frac{d^2w}{dz^2} + (b - z)\,\frac{dw}{dz} - aw = 0, \text{ [1, p. 123, 124]} $$

(2·2) The Whittaker functions $W_{k,m}$ are expressible as linear combination of ${}_1F_1$'s. Two subsidiary solutions of the basic Whittaker equation

$$\frac{d^2W}{dz^2} + \left[-\tfrac{1}{4} + \frac{k}{z} + \frac{\frac{1}{4} - m^2}{z^2}\right] W = 0$$

are, if $2m$ is not an integer,

$$M_{k,m}(z) = z^{m+\frac{1}{2}} e^{-z/2} {}_1F_1\left(\tfrac{1}{2} + m - k ; 2m + 1 ; z\right)$$

and $M_{k,-m}(z)$ [1, p. 127]

3. The following results have been used.

(3·1)
$$\int_0^1 x^{\mu-\frac{1}{2}} (1-x)^{\nu-\frac{1}{2}} M_{k,\mu}(xy)\, M_{\lambda,\nu}\{(1-x)y\}dx$$

$$= \frac{\Gamma(2\mu+1)\,\Gamma(2\nu+1)}{\Gamma(2\mu+2\nu+2)} \cdot M_{k+\lambda,\ \mu+\nu+\frac{1}{2}}(y),$$

$$Re\ \mu > -\tfrac{1}{2}, \quad Re\ \nu > -\tfrac{1}{2}$$ [2, p. 402]

(3·2)
$$\int_0^1 x^{\beta-1} (1-x)^{\sigma-\beta-1} {}_1F_1(\alpha ; \beta ; \lambda xy)\ {}_1F_1(\sigma-\alpha ; \sigma-\beta ; \mu(1-x)y)\, dx$$

$$= \frac{\Gamma(\beta)\,\Gamma(\sigma-\beta)}{\Gamma(\sigma)} \cdot e^{\lambda y} {}_1F_1(\alpha ; \sigma ; (\mu-\lambda)y),$$

$$0 < Re\ \beta < Re\ \sigma$$ [2, p. 402]

(3·3)
$$L_n^{(\alpha)}(x) = \frac{(\alpha+1)_n}{(n)!} {}_1F_1(-n ; \alpha+1 ; x)$$

where $L_n^{(\alpha)}(x)$ is Laguerre polynomial whose Rodrigues formula is given by

$$L_n^{(\alpha)}(x) = \frac{e^x x^{-\alpha}}{(n)!}\left(\frac{d}{dx}\right)^n [e^{-x} x^{n+\alpha}]$$ [3, p. 84]

The operator I_n is defined by

(3·4)
$$In\{f(v)\} = \frac{e^{-v}}{\Gamma(n)} \int_v^1 (y-v)^{n-1} e^y f(y)\, dy,$$

If n is a positive integer, then we have

(3·5)
$$\frac{d}{dv}[e^v I_n\{f(v)\}] = -e^v I_{n-1}\{f(v)\},$$

(3·6)
$$\left(\frac{d}{dv}\right)^n [e^v I_n(f(v))] = (-1)^n e^v f(v),$$

(3·7)
$$I_n\{f(v)\} = 0,\ v = 1 ;\ I_0\{f(v)\} = f(v).$$

4. Theorem (*i*)

If

(*a*) $\mu \geq -\frac{1}{2}, \nu \geq -\frac{1}{2}$,

(*b*) $f(1) = 0$,

(*c*) $\frac{d}{dt} f(t)$ is piecewise continuous on I, then the integral equation

$$\int_t^1 F(u, t)\, y(u, t)\, du = f(t) \tag{4·1}$$

where the integral is taken in the Riemann sense,

$t \in I = \{t : C \leqslant t \leqslant 1\}$, $C > 0$ is a contant and $f(t)$ is defined on I, has the solution

$$y(u, t) = -A \int_u^1 \frac{G(v, u)}{H(v, t)} \cdot d\{f(v)\}, \tag{4·2}$$

where
$$F(u, t) = (u - t)^{\mu - \frac{1}{2}} M_{k,\mu}(u - t)$$
$$G(v, u) = (v - u)^{\nu - \frac{1}{2}} M_{\lambda,\nu}(v - u)$$
$$H(v, t) = (v - t)^{\mu+\nu} M_{k+\lambda,\ \mu+\nu+\frac{1}{2}}(v - t)$$

$$A = \frac{\Gamma(2\mu + 2\nu + 2)}{\Gamma(2\mu + 1)\,\Gamma(2\nu + 1)}$$

Proof:

Putting the value of $y(u, t)$ from (4·2) into the left hand side of (4·1) and changing the order of integration, we find

$$J = -A \int_t^1 \frac{d\{f(v)\}}{H(v, t)} \int_t^v G(v, u)\, F(u, t)\, du \tag{4·3}$$

Now writing $u - t = x(v - t) = xy$, we get

$$\int_t^v G(v, u)\, F(u, t)\, du = \int_t^v (v-u)^{\nu - \frac{1}{2}} M_{\lambda,\nu}(v-u)\,(u-t)^{\mu - \frac{1}{2}} M_{k,\mu}(u-t)\, du \tag{4·4}$$

$$= y^{\mu+\nu} \int_0^1 x^{\mu - \frac{1}{2}} (1-x)^{\nu - \frac{1}{2}} M_{k,\mu}(xy)\, M_{\lambda,\nu}\{(1 - x)y\}\, dx$$

$$= y^{\mu+\nu} A^{-1} M_{k+\lambda,\ \mu + \nu + \frac{1}{2}}(y)$$

after using (3·1),

$$= A^{-1} H(v, t)$$

Therefore, from (4·3) and (4·4), we get

$$J = f(t).$$

Thus the theorem is proved.

5. *Theorem (ii)*

If

(a) $0 < Re\ \beta < Re\ \sigma$,

(b) $f(1) = 0$,

(c) $\frac{d}{dt} f(t)$ is piecewise continuous on I, then the integral equation (4·1) has the solution (4·2)

where
$$F(u, t) = (u - t)^{\beta-1}\ {}_1F_1\left[\alpha\ ;\ \beta\ ;\ \frac{\lambda(u-t)}{\mu-\lambda}\right]$$

$$G(v, u) = (v - u)^{\sigma-\beta-1}\ {}_1F_1\left[\sigma - \alpha\ ;\ \sigma - \beta\ ;\ \frac{\mu(v-u)}{\mu-\lambda}\right]$$

$$H(v, t) = (v - t)^{\sigma-1}\ {}_1F_1(\alpha\ ;\ \sigma\ ;\ v - t).\ e^{\frac{\lambda(v-t)}{\mu-\lambda}}$$

$$A = \frac{\Gamma(\sigma)}{\Gamma(\beta)\,\Gamma(\sigma-\beta)}$$

Proof:

As in (4·3), we get

$$J = -A \int_t^1 \frac{d\{f(v)\}}{H(v,t)} \int_t^v G(v, u)\, F(u,t)\, du \tag{5·1}$$

Now writing $u - t = x\,(v - t) = x\,(\mu - \lambda)\,y$, we get

$$\int_t^v G(v, u)\, F(u, t)\, du \tag{5·2}$$

$$= \int_t^v (v - u)^{\sigma-\beta-1}\ {}_1F_1\left[\sigma - \alpha\ ;\ \sigma - \beta\ ;\ \frac{\mu(v-u)}{\mu-\lambda}\right](u - t)^{\beta-1} \times {}_1F_1\left[\alpha\ ;\ \beta\ ;\ \frac{\lambda\,(u-t)}{\mu-\lambda}\right] du$$

$$= \left[(\mu - \lambda)y\right]^{\sigma-1} \int_0^1 x^{\beta-1}\,(1 - x)^{\sigma-\beta-1}\ {}_1F_1(\alpha\ ;\ \beta\ ;\ \lambda xy) \times {}_1F_1[\sigma - \alpha\ ;\ \sigma - \beta\ ;\ \mu(1 - x)y]\,dx$$

$$= [(\mu-\lambda)y]^{\sigma-1}\, A^{-1}\, e^{\lambda y}\ {}_1F_1[\alpha\ ;\ \sigma\ ;\ (\mu - \lambda)y],$$

after using (3·2),

$$= A^{-1} H(v, t)$$

Therefore from (5·1) and (5·2), the theorem is proved.

6. *Theorem (iii)*

If

(*a*) α is zero or a negative integer and σ is a positive integer,

(*b*) $0 < \beta < \sigma$,

(*c*) $f^{(k)}(1) = 0$ for $0 \leqslant k \leqslant \sigma - \alpha - 1$, and

(*d*) $\left(\frac{d}{dv}\right)^{\sigma-\alpha} \{f(v)\}$ piecewise continuous, then the integral equation

$$e^{\frac{\lambda t}{\mu-\lambda}} \int_t^1 (u-t)^{\beta-1} {}_1F_1\left[\alpha\,;\,\beta\,;\,\frac{\lambda(u-t)}{\mu-\lambda}\right] y(u)\,du = f(t) \tag{6·1}$$

where the integral is taken in the Riemann sense, $t \in I = \{t : C \leqslant t \leqslant 1\}$, $C > 0$ is a constant and $f(t)$ is defined on I, has the solution

$$y(u) = \frac{1}{\Gamma(\beta)\,\Gamma(\sigma-\beta)} \int_u^1 e^{-\frac{\lambda v}{\mu-\lambda}} (v-u)^{\sigma-\beta-1} \times {}_1F_1\left[\sigma-\alpha\,;\,\sigma-\beta\,;\,\frac{\mu(v-u)}{\mu-\lambda}\right] I_{-\alpha}\{F(v)\}dv \tag{6·2}$$

where $F(v) = \left(-\frac{d}{dv}\right)^{\sigma-\alpha} \{f(v)\}$

Proof:

Putting the value of $y(u)$ from (6·2) into the left hand side of the equation (6·1), changing the order of integration and using (5·2), we get

$$J = \frac{1}{\Gamma(\sigma)} \int_t^1 (v-t)^{\sigma-1} {}_1F_1(\alpha\,;\,\sigma\,;\,v-t)\, I_{-\alpha}\{F(v)\}\,dv \tag{6·3}$$

Since α is zero or a negative integer, we have from (3·3),

$${}_1F_1(\alpha\,;\,\sigma\,;\,v-t) = \frac{(-\alpha)!}{(\sigma)_{-\alpha}} L_{-\alpha}^{(\sigma-1)}(v-t) \tag{6·4}$$

$$= \frac{(-\alpha)!}{(\sigma)_{-\alpha}} \cdot \frac{e^{v-t}(v-t)^{-\sigma+1}}{(-\alpha)!} \cdot \left(\frac{d}{dv}\right)^{-\alpha} \left[e^{-v+t}(v-t)^{-\alpha+\sigma-1}\right]$$

$$= \frac{1}{(\sigma)_{-a}} \cdot e^{v}(v-t)^{-\sigma+1}\left(\frac{d}{dv}\right)^{-a}\left[e^{-v}(v-t)^{\sigma-a-1}\right]$$

From (6·3) and (6·4), we get

$$(6\cdot5)\qquad J = \frac{1}{\Gamma(\sigma-a)}\int_t^1 \left[\left(\frac{d}{dv}\right)^{-a}\{(v-t)^{\sigma-a-1}e^{-v}\}\right]\left[e^{v} I_{-a}\{F(v)\}\right]dv$$

After Successive integrations by parts and application of (3·4), (3·5), (3·6) and (3·7), the equation (6·5) becomes

$$(6\cdot6)\qquad J = \frac{1}{\Gamma(\sigma-a)}\int_t^1 (v-t)^{\sigma-a-1}\left(-\frac{d}{dv}\right)^{\sigma-a}\{f(v)\}dv$$

Further successive integrations by parts and the application of the conditions (c), yield $J = f(t)$. Thus the theorem is proved.

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References

1. Rainville, E. D. Special functions. *The Macmillan Company, New York,* 1963.
2. Erdélyi, A. Tables of integral transforms. Vol. **2**, *McGraw-Hill, New York,* 1954.
3. Magnus, W. and Oberthettinger, F. Translated by J. Wermer. *Special functions of Mathematical Physics, New York,* 1949.
4. Srivastava, K. N. Integral equations involving a confluent hypergeometric function as kernel. *Journal D' Analyse Mathematige,* **13**, *Jerusalem,* 1964.
5. Ta Li. A new class of integral transform. *Proc. Amer. Math. Soc.*, **11**, 290–298, 1960.
6. Bushman, R. G. An inversion integral for a Legendre transformation. *Amer. Math. Monthly,* **69**, 288–289, 1962.
7. Srivastava, K. N. A class of integral equations involving Ultra spherical polynomials as kernel. *Proc. Amer. Math. Soc.*, **14**, 932–940, 1963.
8. Srivastava, K. N. On some integral transforms. *Mathematica Japonicae of University of Osaka Prefecture,* **4**(3,4).
9. Widder, D. V. The inversion of a convolution transform whose kernel is a Laguerre polynomial. *Amer. Math. Monthly,* **70**, 1963.
10. Srivastava, K. N. On some integral equations involving Jacobi polynomial. *Department of Mathematics, College of Engineering University of Osaka Prefecture, Mozu, Sakai, Japan.*

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H-Function and Heat Production in a Cylinder

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Abstract

In this paper, we have considered the diffusion of heat in an infinitely long cylinder of radius a when there are sources of heat within it, which lead to an axially symmetrical temperature distribution. The function, governing the heat source has been assummed to have the form of $(k/K)\,f(r)\,g(t)$, where k and K are diffusivity and conductivity of the meterial respectively. We have characterised the heat source by the behaviour of $g(t)$, which has been taken in a most general form, involving H-function given by Fox, so as to yield, in turn, various interesting heat sources as its particular cases.

1. Cases in which heat is produced in solids are important in technical applications [4, pp. 12–13]. Heat may be produced by the passage of an electric current, dielectric or induction heating, radioactive decay, absorption from radiation, mechanical generation in viscous or plastic flow, chemical reaction, hydration of cement, and the ripening of apples. Nuclear reactors and space research also give rise to different problems of heat transfer.

Here, we consider the diffusion of heat in a cylinder of radius a when there are sources of heat within it, which lead to an axially symmetrical temperature distribution. In this case the fundamental equation is of the form [11, p. 202] :

$$\frac{\partial u}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \theta(r, t), \tag{1·1}$$

if we assume that the rate of generation of heat is independent of the temperature and that the cylinder is infinitely long, so that the variation with z may be neglected. We shall, in addition, suppose that the surface $r = a$ is maintained at zero temperature and the initial distribution of temperature is also zero.

We further suppose that

$$\theta(r, t) = (k/K)\,f(r)\,g(t), \tag{1·2}$$

where k is the diffusivity and K the conductivity of the meterial.

In this paper, we have chosen throughout.

$$f(r) = r^{2\rho+2\delta-4}\left(\frac{a^2}{r^2} - 1\right)^{\sigma-1} {}_2F_1\left[\lambda + \sigma - \rho,\ \mu + \sigma - \rho,\ \sigma\ ;\ 1 - \frac{a^2}{r^2}\right] \tag{1·3}$$

alongwith $g(t)$, by behaviour of which we shall characterise the heat source.

2. Finite Hankel Transform. Let the finite Hankel transform of $f(r)$ be [11, p. 83]

$$(2\cdot1) \qquad J[f(r)] = \int_0^a r f(r) J_0(r\,\xi_i)\,dr = \bar{f}_j(\xi_i),$$

then taking $m = 0$, $n = 1$, $p = 2$, $q = 0$, $a_1 = a_2 = 1 - d$, $\alpha = 4/(a^2\,\xi_i^2)$, replacing x by $(1/y)$, y by (r^2/a^2) and using the identities [5, p. 209 (9) and p. 216 (3)], the known integral [6, p. 421 (13)] yields

$$(2\cdot2) \qquad J\left[r^{2\rho+2d-4}\left(\frac{a^2}{r^2}-1\right)^{\sigma-1}\,{}_2F_1\left(\lambda+\sigma-\rho,\ \mu+\sigma-\rho\,;\,\sigma\,;\,1-\frac{a^2}{r^2}\right)\right]$$
$$= 2^{2d-1}\,a^{2\rho-2}\,\Gamma(\sigma)\,\xi_i^{-2d}\,G^{2,\,1}_{4,\,2}\left[\frac{4}{a^2\,\xi_i^2}\,\middle|\,\begin{matrix}1-d,\ 1-d,\ \lambda+\mu+\sigma-\rho,\ \rho\\ \lambda,\ \mu\end{matrix}\right],$$

where $Re(\sigma) > 0$, $Re(\lambda) \geq Re(\mu) > -Re(d)$, $Re(d+\lambda) < \frac{3}{4}$, $Re(d+\mu) < \frac{3}{4}$ and ξ_i is the root of the transcedental equation :

$$(2\cdot3) \qquad J_0(a\,\xi_i) = 0.$$

By virtue of the inversion theorem [11, p. 83] we have

$$(2\cdot4) \qquad r^{2\rho+2d-4}\left(\frac{a^2}{r^2}-1\right)^{\sigma-1}\,{}_2F_1\left[\lambda+\sigma-\rho,\ \mu+\sigma-\rho\,;\,\sigma\,;\,1-\frac{a^2}{r^2}\right]$$
$$= 2^{2d}\,a^{2\rho-2}\,\Gamma(\sigma)\,\Sigma_i\,\xi_i^{-2d}\,G^{2,\,1}_{4,\,2}\left[\frac{4}{a^2\,\xi_i^2}\,\middle|\,\begin{matrix}1-d,\ 1-d,\ \lambda+\mu+\sigma-\rho,\ \rho\\ \lambda,\ \mu\end{matrix}\right]\frac{J_0(r\,\xi_i)}{[J_1(a\,\xi_i)]^2},$$

where the summation is taken over all the positive roots of the equation (2·3).

The result (2·4) will be useful in the verification of the solution.

3. Solution of the problem. We apply the finite Hankel transform to obtain the solution of (1·1). Its solution will be obtained as [11, p 203 (16)] :

$$(3\cdot) \qquad u(r,\,t) = 2^{2d}\,\frac{k}{K}\,a^{2\rho-4}\,\Gamma(\sigma)\,\Sigma_i\,\xi_i^{-2d}\,G^{2,\,1}_{4,\,2}\left[\frac{4}{a^2\,\xi_i^2}\,\middle|\,\begin{matrix}1-d,\ 1-d,\ \lambda+\mu+\sigma-\rho,\ \rho\\ \lambda,\ \mu\end{matrix}\right]\times$$
$$\times\,\frac{J_0(r\,\xi_i)}{[J_1(a\,\xi_i)]^2}\cdot h(\xi_i,\,t),$$

where the sum is taken over all the positive roots of the equation (2·3) and

$$(3\cdot2) \qquad h(\xi_i,\,t) = \int_0^t g(\tau)\,e^{-k\,\xi_i^2\,(t-\tau)}\,d\tau.$$

We shall evaluate (3·2) for a heat source of very general and interesting nature. Various heat source may follow as its particular cases.

4. Heat source of General Character. Let us choose

$$(4\cdot1) \qquad g(\tau) = g_0\tau^{\gamma-1}\,(t-\tau)^{\rho-1}\,e^{-z\tau}\,{}_2F_1\left[\alpha,\ \beta;\ \gamma;\ \frac{\tau}{t}\right]H^{m,\,n}_{p,\,q}\left[c\left(1-\frac{\tau}{t}\right)^{\frac{\delta}{s}}\,\middle|\,\begin{matrix}\{(a_p,\alpha_p)\}\\ \{(b_q,\beta_q)\}\end{matrix}\right],$$

where, for the sake of brevity, $H^{m,\,n}_{p,\,q}\left[x\,\middle|\,\begin{matrix}\{(a_p,\,\alpha_p)\}\\ \{(b_q,\,\beta_q)\}\end{matrix}\right]$ stands for

$H_{p,q}^{m,n}\left[x \,\middle|\, \begin{matrix}(a_1, \alpha_1), \ldots, (a_p, \alpha_p)\\ (b_1, \beta_1), \ldots, (b_q, \beta_q)\end{matrix}\right]$, a notation given by Gupta [8, p. 98] to represent the H-function, introduced by Fox [7, p. 408] as

$$H(x) = \frac{1}{2\pi i}\int_T \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j s)}\, x^s\, ds,$$

where x is not equal to zero and empty product is interpreted as unity ; p, q, m and n are integers satisfying $1 \leqslant m \leqslant q$, $0 \leqslant n \leqslant p$; α_j $(j = 1, \ldots, p)$, β_j $(j = 1, \ldots, q)$ are positive numbers and a_j $(j = 1, \ldots, p)$ b_j $(j = 1, \ldots, q)$, are complex numbers such that no pole of $\Gamma(b_h - \beta_h s)$ $(h = 1, \ldots, m)$ coincides with any pole of $\Gamma(1 - a_i + \alpha_i s)$ $(i = 1, \ldots, n)$, *i.e.*

$$\alpha_i(b_h + \nu) \neq (a_i - \eta - 1)\,\beta_h \qquad (4\cdot2)$$

$$(\nu, \eta = 0, 1, \ldots ; h = 1, \ldots, m ; i = 1, \ldots, n).$$

Further, the contour T runs from $\sigma - i\infty$ to $\sigma + i\infty$ such that the points $s = (b_h + \nu)/\beta_h$ $(h = 1, \ldots, m ; \nu = 0, 1, \ldots,)$, which are poles of $\Gamma(b_h - \beta_h s)$ $(h = i, \ldots, m)$ lie on the right and the points $s = (a_i - \eta - 1)/\alpha_i$ $(i = 1, \ldots, n ; \eta = 0, 1, \ldots,)$, which are poles of $\Gamma(1 - a_i + \alpha_i s)$ $(i = 1, \ldots, n)$ lie to the left of T. Such a contour is possible on account of (4·2).

According to Braaksma [3, p. 278] $H(x) = 0\ (|x|^{\alpha})$ for small x,

where $\sum_1^p \alpha_j - \sum_1^q \beta_j \leqslant 0$ and $\alpha = \min. Re(b_h/\beta_h)$ $(h = 1, \ldots, m)$ and $H(x) = 0\ (|x|^{\beta})$

for large x, where $\sum_1^p \alpha_j - \sum_1^q \beta_j < 0$, $\sum_1^n \alpha_j - \sum_{n+1}^p \alpha_j + \sum_1^m \beta_j - \sum_{m+1}^q \beta_j \equiv \lambda > 0$,

$|\arg z| < \frac{1}{2}\lambda\pi$ and $\beta = \max. \ Re\left(\frac{a_i - 1}{\alpha_i}\right)$ $(i = 1, \ldots, n)$.

Substituting the above $g(\tau)$ from (4·1) in (3·2) and then replacing τ by $t\tau$ we get

$$h(\xi_i, t) = g_0\, t^{\gamma+\rho-1}\, e^{-k\xi_i^2 t}\int_0^1 \tau^{\gamma-1}(1-\tau)^{\rho-1}\, e^{-t(z-k\xi_i^2)\tau}\, {}_2F_1[\alpha, \beta ; \gamma ; \tau] \times \qquad (4\cdot3)$$

$$\times\ H_{p,q}^{m,n}\left[c(1-\tau)\frac{\delta}{s}\,\middle|\, \begin{matrix}\{(a_p, \alpha_p)\}\\ \{(b_q, \beta_q)\}\end{matrix}\right] d\tau.$$

Evaluating the integral in (4·3) with the help of a result, recently given by the author[10] as

$$\text{“}\int_0^1 x^{\gamma-1}(1-x)^{\rho-1} e^{-zx}\, {}_2F_1[\alpha, \beta; \gamma; x]\, H_{p,q}^{m,n}\left[\xi(1-x)^{\delta/s}\,\middle|\,\begin{matrix}\{(a_p, \alpha_p)\}\\ \{(b_q, \beta_q)\}\end{matrix}\right] dx$$

$$= (2\pi)^{\frac{(1-s)(m+n-\frac{1}{2}p-\frac{1}{2}q)}{s}} \delta^{\sum_1^q b_j - \sum_1^p a_j + \frac{1}{2}(p-q)+1} \delta^{-\gamma} e^{-z}\, \Gamma(\gamma) \sum_{\omega=0}^{\infty} \frac{z^\omega}{\omega!} \times$$

$$\times H_{sp+2\delta,\, sq+2\delta}^{sm,\, sn+2\delta}\left[(\xi s^\eta)^s \,\middle|\, \begin{matrix}(\triangle(\delta, 1-\rho-\omega+\left|{}^{\circ}_{\alpha+\beta-\gamma}\right|), 1), \{(\triangle(s, a_p), \alpha_p)\}\\ \{(\triangle(s, b_q), \beta_q)\}, (\triangle(\delta, 1-\rho-\gamma-\omega+\left|{}^{\alpha}_{\beta}\right|), 1)\end{matrix}\right],$$

where δ and s are positive integers; $\{(\triangle(s, a_p), \alpha_p)\}$ stands for the set of parameters $(\triangle(s, a_1), \alpha_1), \ldots, (\triangle(s, a_p), \alpha_p)$;

$(\triangle(s, a_1), \alpha_1)$ denotes $\left(\frac{a_1}{s}, \alpha_1\right), \left(\frac{a_1+1}{s}, \alpha_1\right), \ldots, \left(\frac{a_1+s-1}{s}, \alpha_1\right)$;

$(\triangle(\delta, a+\left|{}^{\alpha}_{\beta}\right|), 1)$ represents $(\triangle(\delta, a+\alpha), 1), (\triangle(\delta, a+\beta), 1)$;

$\sum_1^p \alpha_j - \sum_1^q \beta_j \equiv \eta \leqslant 0$, $\sum_1^n \alpha_j - \sum_{n+1}^p \alpha_j + \sum_1^m \beta_j - \sum_{m+1}^q \beta_j \equiv \lambda > 0$, $|\arg \xi| < \frac{1}{2}\lambda\pi$.

$Re(\gamma) > 0$, $Re\left(\rho+\frac{\delta}{s}\cdot\frac{b_h}{\beta_h}\right) > 0$ and $Re\left(\gamma+\rho-\alpha-\beta+\frac{\delta}{s}\cdot\frac{b_h}{\beta_h}\right) > 0$ $(h=1, \ldots m)$” we get

$$(4\cdot4)\quad h(\xi_i, t) = g_0(2\pi)^{\frac{(1-s)(m+n-\frac{1}{2}p-\frac{1}{2}q)}{s}} \delta^{\sum_1^q b_j - \sum_1^p a_j + \frac{1}{2}(p-q)+1} \delta^{-\gamma}\, \Gamma(\gamma)\, e^{-zt} \times$$

$$\times t^{\gamma+\rho-1} \sum_{\omega=0}^{\infty} \frac{(z-k\xi_i)^\omega\, t^\omega}{\omega!} \times$$

$$\times H_{sp+2\delta,\, sq+2\delta}^{sm,\, sn+2\delta}\left[(cs^\eta)^s \,\middle|\, \begin{matrix}(\triangle(\delta, 1-\rho-\omega+\left|{}^{\circ}_{\alpha+\beta-\gamma}\right|), 1), \{(\triangle(s, a_p), \alpha_p)\}\\ \{(\triangle(s, b_q), \beta_q)\}, (\triangle(\delta, 1-\rho-\gamma-\omega+\left|{}^{\alpha}_{\beta}\right|), 1)\end{matrix}\right],$$

where δ and s are positive integers, $\sum_1^p \alpha_j - \sum_1^q \beta_j \equiv \eta \leqslant 0$,

$\sum_1^n \alpha_j - \sum_{n+1}^p \alpha_j + \sum_1^m \beta_j - \sum_{m+1}^q \beta_j \equiv \lambda > 0$, $|\arg c| < \frac{1}{2}\lambda\pi$, $Re(\gamma) > 0$,

$Re\left(\rho+\frac{\delta}{s}\frac{b_h}{\beta_h}\right) > 0$, $Re\left(\gamma+\rho-\alpha-\beta+\frac{\delta}{s}\cdot\frac{b_h}{\beta_h}\right) > 0$ $(h = 1, \ldots, m)$.

Hence from (3·1) and (4·4) we obtain

$$(4\cdot5)\qquad u(r, t) = g_0 2^{2d}\frac{k}{K} a^{2\rho-4}\, \Gamma(\sigma)\, \Gamma(\gamma)\, (2\pi)^{(1-s)(m+n-\frac{1}{2}p-\frac{1}{2}q)} \times$$

$$\times s^{\sum_1^q b_j - \sum_1^p a_j + \frac{1}{2}(p-q)+1}\, \delta^{-\gamma_i \gamma+\rho-1}\, e^{-z^t}\, \Sigma_i\, \xi_i^{-2\delta} \times$$

$$\times G^{2,1}_{4,2}\left[\frac{4}{a^2\xi_i^2}\,\middle|\,\begin{matrix}1-d,\ 1-d,\ \lambda+\mu+\sigma-\rho,\ \rho\\ \lambda,\ \mu\end{matrix}\right]\frac{J_0(r\,\xi_i)}{[J_1(a\,\xi_i)]^2}\sum_{\omega=0}^{\infty}\frac{(z-k\,\xi_i)^{\omega_i\omega}}{\omega\,!}\times$$

$$\times H^{sm,\ sn+2\delta}_{sp+2\delta,\ sq+2\delta}\left[(cs^{\gamma_i})^s\,\middle|\,\begin{matrix}(\Delta(\delta,\ 1-\rho-\omega+\left|\begin{matrix}\circ\\ \alpha+\beta-\gamma\end{matrix}\right|),\ 1),\ \{(\Delta(s,\ a_p),\ \alpha_p)\}\\ \{(\Delta(s,\ b_q),\ \beta_q)\},\ (\Delta(\delta,\ 1-\rho-\gamma-\omega+\left|\begin{matrix}\alpha\\ \beta\end{matrix}\right|),\ 1)\end{matrix}\right].$$

Obviously $u(r, 0) = 0$.

5. Heat sources of various characters. The H-function, involved in $g(\tau)$ is of a very general nature and yields, as particular cases[10], various functions *viz.* Meijer's G-function [5, p. 207], Macrobert's E-function [5, p. 203], Bessel functions, Wright's generalised hypergemetric function [12, p. 287], Maitland's Bessel function [13, p. 257], confluent hypergeometric functions, Gauss' hypergeometric functions and exponential function, etc. Therefore, the heat sources of various known and unknown characters may follow as particular cases of $g(\tau)$, defined in (4·1).

By using the identity :

$$H^{m,\ n}_{p,\ q}\left[x\,\middle|\,\begin{matrix}\{(a_p,\ 1)\}\\ \{(b_q,\ 1)\}\end{matrix}\right] \equiv G^{m,\ n}_{p,\ q}\left[x\,\middle|\,\begin{matrix}a_1,\ \ldots,\ a_p\\ b_1,\ \ldots,\ b_q\end{matrix}\right],$$

where $G^{m,\ n}_{p,\ q}\left[x\,\middle|\,\begin{matrix}a_1,\ \ldots,\ a_p\\ b_1,\ \ldots,\ b_q\end{matrix}\right]$ is Meijer's G-function [5, p. 207], and another known identity [6, p. 437 (11)], if we reduce the H-function involved in (4·1) into the form ${}_2F_1[A, B\,;\, C\,;\, x]$, we arrive at the heat source recently given by Bajpai[1] on putting $A = B = 0$; and further with $\alpha = \beta = 0$ and by application of the convolution theorem for Laplace transform a known heat source due to Bhonsle [2, p. 86] is obtained.

6. Verification of the Solution. From (3·1), on using the known results [9, p. 100 (5·2·4 and 5·2·5)], we get

$$(6\cdot1)\qquad \frac{k}{K}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = -\frac{k^2}{K}\,2^{2d}\,a^{2\rho-4}\,\Gamma(\sigma)\,\Sigma_i\,\xi_i^{-2d}\times$$

$$\times G^{2,1}_{4,2}\left[\frac{4}{a^2\xi_i^2}\,\middle|\,\begin{matrix}1-d,\ 1-d,\ \lambda+\mu+\sigma-\rho,\ \rho\\ \lambda,\ \mu\end{matrix}\right]\frac{\xi_i^2 J_0(r\,\xi_i)}{[J_1(a\,\xi_i)]^2}\int_0^t g(\tau)\,e^{-k\,\xi_i^2\,(t-\tau)}\,d\tau.$$

From (1·2) and (2·4) we have

$$(6\cdot2)\qquad \theta(r, t) = 2^{2d}\,\frac{k}{K}\,a^{2\rho-4}\,\Gamma(\sigma)\,\Sigma_i\,\xi_i^{-2d}\,G^{2,1}_{4,2}\left[\frac{4}{a^2\xi_i^2}\,\middle|\,\begin{matrix}1-d,\ 1-d,\ \lambda+\mu+\sigma-\rho,\ \rho\\ \lambda,\ \mu\end{matrix}\right]\times$$

$$\times\frac{J_0(r\,\xi_i)}{[J_1(a\,\xi_i)]^2}\cdot g(t).$$

Also from (3·1) we obtain

$$(6\cdot3)\qquad \frac{\partial u}{\partial t} = 2^{2d}\,\frac{k}{K}\,a^{2\rho-4}\,\Gamma(\sigma)\,\Sigma_i\,\xi_i^{-2d}\,G^{2,1}_{4,2}\left[\frac{4}{a^2\xi_i^2}\,\middle|\,\begin{matrix}1-d,\ 1-d,\ \lambda+\mu+\sigma-\rho,\ \rho\\ \lambda,\ \mu\end{matrix}\right]\times$$

$$\times \frac{J_0(r\,\xi_i)}{[J_1(a\,\xi_i)]^2}\left[\, g(t) - k\,\xi_i^2 \int_0^t g(\tau)\; e^{-k\,\xi_i^2\,(t-\tau)} d\tau \,\right].$$

Substituting the above values from (6·1), (6·2) and (6·3) in (1·1), we notice that the equation is satisfied.

The boundary condition $u(a, t) = 0$ is satisfied because $J_0\,(a\;\xi_i)$, which is present in every term of $u(a, t)$, is zero. The initial condition is satisfied, because $h(\xi, 0) = 0$.

Flux at any point will be given by

$$(6\cdot4) \quad -K\frac{\partial u}{\partial r} = 2^{2d}\,k\,a^{2\rho-4}\,\Gamma(\sigma)\,\Sigma_i\,\xi_i^{-2d}\; G^{2,1}_{4,2}\left[\frac{4}{a^2\,\xi_i^2}\middle|\begin{matrix}1-d,\ 1-d,\ \lambda+\mu+\sigma-\rho,\ \rho\\ \lambda,\ \mu\end{matrix}\right]\times$$

$$\times \frac{\xi_i\,J_0'\,(r\;\xi_i)}{[J_1(a\;\xi_i)]^2}\int_0^t g(\tau)\,e^{-k\,\xi_i^2\,(t-\tau)}\,d\tau.$$

It can easily be shown that (3·1) converges uniformly when $t > 0$ and so the function $u(r, t)$, represented by it, is continuous when $0 \leqslant r \leqslant a$.

The term by term differentiations are justified because (6·1), (6·3) and (6·4) are uniformly convergent when $t > 0$, $0 \leqslant r \leqslant a$.

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References

1. Bajpai, S. D. Associated Legendre functions and heat production in a cylinder. (*To appear*).
2. Bhonsle, B. R. Jacobi polynomials and heat production in a cylinder. *Math. Japonica*, **11**, 83–90, 1966.
3. Braasksma, B. L. J. Asymptotic expansion and analytic continuations for a class of Barnes integrals. *Compos. Math.*, **15**, 239–341, 1963.
4. Carslaw, H. S. and Jaeger, J. C. Conduction of heat in solids, Ed. II. *Oxford Univ. Press, London*, 1959.
5. Erdelyl, A. Higher transcedental functions, *McGraw-Hill, N. Y.*, **1**, 1953.
6. Erdelyl A. Tables of integral transforms, *McGraw Hill, N. Y.*, **2**, 1954.
7. Fox, Charles. The G- and H-functions as symmetrical Fourier kernels. *Trans. Amer. Math. Soc.*, **98**, 395–429, 1961.
8. Gupta, K. C. On the H-function. *Anneles de la societe Scientifique de Bruxelles*, **T79 II**, 97–106, 1965.
9. Lebedev, N. N. Special functions and their applications. *Prentice-Hall, INC, Englewood Cliff, N. J.*, 1965.
10. Sharma, O. P. Some finite and infinite integrals involving H-function and Gauss' hypergeometric functions. *Collectanea Mathematica*, **17**, 197-209, 1965.
11. Sneddon, I. N. Fourier transform. *McGraw–Hill, N. Y.* 1951.
12. Wright, W. M. The asymptotic expansion of the generalised hypergeometric function. *J. Lond. Math. Soc.*, **10**, 287, 1935.
13. Wright, W. M. The asymptotic expansion of the generalised Bessel function. *Proc. Lond. Math. Soc.*, **38**, 257, 1935.

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Muskhelishvili's technique applied to two involved boundaries, for the torsion of beams

By

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Abstract

Two cases are studied, namely :

(*i*) The torsion of a beam whose boundary consists of the parabola $y^2 = x$. and the symmetric curve $r = 2\cos\theta/3$ $\left(|\theta| = \frac{3\pi}{4}\right)$ shown in figure (*a*)

(*ii*) The torsion of a beam whose boundary consists of three pieces, the real axis from $4a$ to a, the parabole $\gamma = \frac{2a}{1+\cos\theta}\left(-\frac{\pi}{2} \leqslant \theta \leqslant 0\right)$ and the cardioid $\gamma = 2a(1+\cos\theta)$ (θ lying between $-\pi/2$ and 0) shown in figure (*a'*).

The functions, mapping these boundaries on the unit circle in the ρ – plane are respectively.

$$z = \frac{2\sqrt{i(\rho^2-1)} + 2\rho(1+i)}{2\sqrt{i(\rho^2-1)} \cdot (\rho+i)(1+i)}$$

and

$$z = \frac{2a\{\sqrt{\rho-1} + \sqrt{i}\sqrt{\rho+1}\}}{\sqrt{\rho-1} + i\sqrt{(\rho-i)(i-1)}}$$

The complex torsion functions are respectively

$$F(\rho) = 2i + \frac{\sqrt{i}(1+i)}{i}\left\{\frac{2i\sqrt{\rho^2-1} + \sqrt{2}}{\rho-i}\right\}$$

and

$$F(\rho) = 4a^2 i + \frac{4a^2\rho}{\rho+1} + \frac{8a^2\sqrt{i}\sqrt{\rho^2-1}}{\rho+1}.$$

The calculation involves the use of the principal values of the integrals as the singularities and branch points fall on the boundary of the unit circle.

1. *Torsion of a beam whose boundary is part of the parabola* $y^2 = -x$ *and the symmetric curve* $\gamma = 2\cos\theta/3$

Consider a region bounded by the parabola $y^2 = -x$ and the curve $\gamma = 2\cos\theta/3$ where $|\theta| < 3\pi/4$

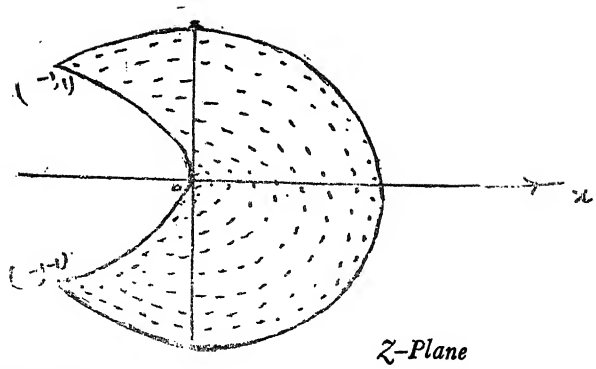


Fig. (*a'*)

Z–Plane

The transformation

$$Z = W + W^2 \tag{1·1}$$

takes the region in figure (*a*) conformally on the semi-circle in the W-plane ($|W| < 1$ Real $W > 0$) Fig. (*b*).

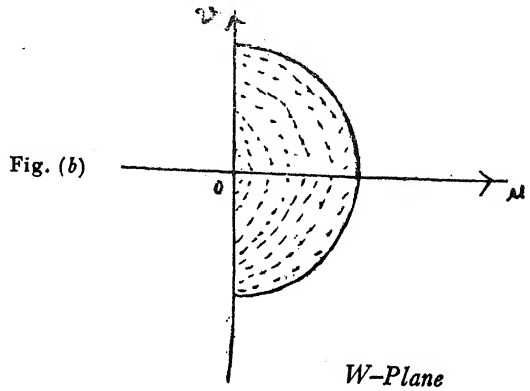


Fig. (*b*)

W–Plane

Again by

$$W_1 = \left(\frac{W - i}{W + i}\right)^2 \tag{1·2}$$

the semi-circle in the W–plane is confirmally mapped on to the upper half of the W_1 – plane.

Ultimately

$$\rho = \frac{i - W_1}{i + W_1} = \frac{i - \left(\frac{W - i}{W + i}\right)^2}{i + \left(\frac{W - i}{W + i}\right)^2} \tag{1·3}$$

maps the upper half of the W-plane conformally on to the unit circle in the ρ – plane.

Combination of (1·1), (1·2) and (1·3) leads to the final mapping of the region in figure (*a*) onto the unit circle in the ρ – plane.

Namely :

$$Z = \omega(\rho) = \frac{2\sqrt{i(\rho^2 - 1)} + 2\rho(1 + i)}{2\sqrt{i(\rho^2 - 1)} - (\rho + i)(1 + i)} \tag{1·4}$$

Now :

$$\omega(\sigma) = \frac{2\sqrt{i(\sigma^2-1)} + 2\sigma(1+i)}{2\sqrt{i(\sigma^2-1)} - (\sigma+i)(1+i)} \tag{1·5}$$

Where $\sigma = e^{i\theta}$ on the unit circle

And

$$\overline{\omega}(\overline{\sigma}) = \frac{2\sqrt{i(\sigma^2-1)} + 2(1-i)}{2\sqrt{i(\sigma^2-1)} + (\sigma+i)(+\overline{i})} \tag{1·6}$$

Muskhelishvili's torsion function is given by

$$F(\rho) = \frac{1}{2\pi}\int \frac{\omega(\sigma)\,\overline{\omega}(\overline{\sigma})}{\sigma-\rho}\,d\sigma \tag{1·7}$$

In the present case (1·7) becomes

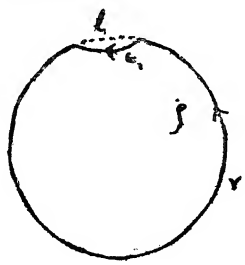
$$F(\rho) = \frac{1}{2\pi}\int \frac{4i(\sigma^2-1) + 4\sqrt{i(\sigma^2-1)}\,(\sigma-i)(1+i) + 8\sigma}{2i(\sigma-i)^2}\cdot\frac{d\sigma}{\sigma-\rho} \tag{1·8}$$

which can be split up into three integrals, studied in detail as under :

$$I_1 = \frac{1}{2\pi}\int \frac{4i(\sigma^2-1)}{2i(\sigma-i)^2}\,\frac{d\sigma}{\sigma-\rho} \tag{1·9}$$

Here $\sigma = i$ is a double pole lying on the contour of integration necessitating the indentation as shown in the figure below :

Integral (1·9) can be written as



$$\int_{\gamma-l} + \int_{\varepsilon_1} = 2\pi i \times (\text{residue at } \rho)$$

where l is the portion of the circle γ shown dotted.

$$\underset{\varepsilon_1\to 0}{\text{Limit}} \int_{\varepsilon_1} = \frac{-i\pi(-2i\rho)}{(i-\rho)^2}$$

so that the principal value of the integral (1·9) turns out to be

$$\frac{2i(\rho^2-1) + 2\rho}{(\rho-i)^2} \tag{1·10}$$

On the similar grounds the 2nd integral given by

$$I_2 = \frac{1}{2\pi}\int \frac{4\sqrt{i}\sqrt{\sigma^2-1}\,(1+i)(\sigma-i)}{2i(\sigma-i)^2}\,\frac{d\sigma}{\sigma-\rho} \tag{1·11}$$

reduces to

$$\frac{\sqrt{i}\,(1+i)}{i}\left[\frac{2i\sqrt{\rho^2-1} + \sqrt{2}}{\rho-i}\right. \tag{1·12}$$

The third integral

$$I_3 = \frac{1}{2\pi} \int \frac{\bar{\sigma}\, \sigma}{2\, i\, (\sigma - i)^2} \cdot \frac{d\sigma}{\sigma - \rho} \tag{1·13}$$

which after a few simple calculations reduces to

$$\frac{2\,\rho}{(\rho - i)^2} \tag{1·14}$$

Adding (1·10), (1·12) and (1·14)

$$F(\rho) = 2\, i + \frac{\sqrt{i}\,(1 + i)}{i} \left\{ \frac{2\, i \sqrt{\rho^2 \cdot 1 + \sqrt{2}}}{\rho - i} \right\} \tag{1·15}$$

with the above complex torsion function the torsion problem stands completely solved.

2. *Torsion of a beam with the Cross-section comprising of a straight segment, a parabola and a cardioid* :

We consider a region bounded by the real axis extending from $4a$ to a and the parabola $R = \frac{2a}{1 + \cos\theta}$ with $\left(-\frac{\pi}{2} \leqslant \theta \leqslant 0 \right)$ and the cardioid $R = 2a\,(1 + \cos\theta)$ where θ changes from $-\pi/2$ to 0.

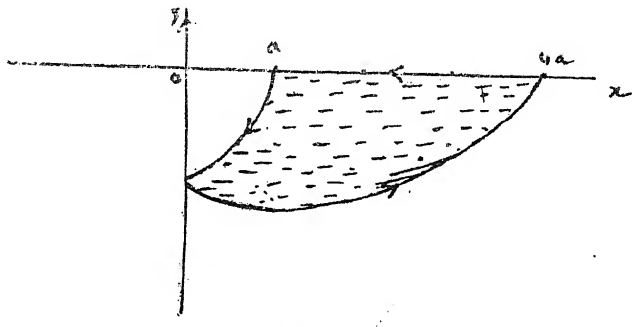


Fig. (*a''*)

Z-plane

The transformation

$$Z = \frac{4a}{(W + 1)^2} \tag{2·1}$$

transforms conformally in figure (a') on to the quadrant in the W-plane figure (b')

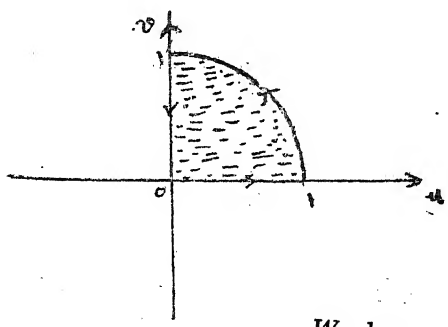


Fig. (*b'*)

W-plane

The parabola in the ζ-plane is transformed to a quadrant $|W| = 1$ $(0 \leqslant \theta \leqslant \pi/2)$ in the W-plane Fig. (b') and the cardioid in the ζ-plane to the imaginary axis in the W-plane.

Again the transformation

$$W' = \left(\frac{1 + W^2}{1 - W^2}\right)^2 \tag{2·2}$$

maps the quadrant in the W-plane conformally on to the upper half of W'-plane.

Finally the transformation

$$\rho = \frac{i - \left(\frac{1 + \omega^2}{1 - W^2}\right)^2}{i + \left(\frac{1 + W^2}{1 - W^2}\right)^2} \tag{2·3}$$

maps the upper half of the W'-plane conformally to the unit circle γ in the ρ-plane.

Combination of (2·1), (2·2) and (2·3) leads to

$$\zeta = \omega(\rho) = \frac{2a\{\sqrt{\rho - 1} + \sqrt{i}\sqrt{\rho + 1}\}}{\sqrt{\rho - 1} + \sqrt{i}\sqrt{(\rho - i)(i - 1)}} \tag{2·4}$$

as the required mapping function.

Now

$$W(\sigma) = \frac{2a\{\sqrt{\sigma - 1} + \sqrt{i}\sqrt{\sigma + 1}\}}{\sqrt{\sigma - 1} + i\sqrt{(\sigma - i)(i - 1)}} \tag{2·5}$$

where σ is a point on the unit circle.

$$\bar{\omega}(\bar{\sigma}) = \frac{2a\{\sqrt{\sigma - 1} + \sqrt{i}\sqrt{\sigma + 1}\}}{\sqrt{\sigma - 1} - i\sqrt{(\sigma - i)(i - 1)}} \tag{2·6}$$

For the complex torsion function we have the expression

$$F(\rho)\ \frac{1}{2\pi}\int \frac{\omega(\sigma)\,\bar{\omega}(\bar{\sigma})}{\sigma - \rho} \tag{2·7}$$

In the problem under-study we have

$$F(\rho) = \frac{4a^2}{2\pi}\int \frac{\sigma - 1}{i(\sigma + 1)}\,\frac{d\sigma}{\sigma - \rho} + \frac{4a^2}{2\pi}\int \frac{d\sigma}{\sigma - \rho} + \frac{8a^2}{2\pi}\int \frac{\sqrt{i}\sqrt{\sigma^2 - 1}}{i(\sigma + 1)}\,\frac{d\sigma}{\sigma - \rho} \tag{2·8}$$

The three integrals of (2·8) are evaluated on the same lines as in the previous problem (principal) values being taking at the singularities lying on the contour.

It can easily be seen that $F(\rho)$ turns out to be given by

$$F(\rho) = 4a^2 i + \frac{4a^2\rho}{\rho + 1} + \frac{8a^2\sqrt{i}\sqrt{\rho^2 - 1}}{\rho + 1} \tag{2·9}$$

This completely solves the torsion problem of the considered boundary.

References

1. Muskhelishvili, N. I. Some basic problems of the mathematical theory of elasticity. *P. Noordhoff, Ltd., Groningen–Holland*, 1953.
2. Volkovyskii, L. I., Lunts, G. L. Aramanovich, I. G. A collection of problems on Complex analysis. *Pergamon Press, Oxford, Edinburgh, Pan's*, 1965.

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On Meijer-Laplace Transform of Two Variables

By

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Abstract

In this paper we have obtained some properties of Meijer-Laplace Transform of two variables involving chain of Meijer-Laplace and Laplace transforms.

1. Introduction

The Meijer-Laplace transform is defined [1, p. 57] by means of the integral equation in the form

$$(1\cdot1)\qquad F(p) = p\int_0^\infty G^{m+1,\,0}_{m,\,m+1}\left(px\,\middle|\,\begin{matrix}\xi_1+\alpha_1, \ldots, \xi_m+\alpha_m\\ \xi_1, \ldots, \xi_m, \mu\end{matrix}\right) f(x)\,dx,\ R(p) > 0.$$

we shall denote it symbolically by

$$F(p) = G[f(x) : \alpha_m, \xi_m, \mu].$$

The Meijer-Laplace transform of two variables has been introduced by the author[7] in the form

$$(1\cdot2)\qquad F(p, q) = pq\int_0^\infty\int_0^\infty G^{m+1,\,0}_{m,\,m+1}\left(px\,\middle|\,\begin{matrix}\xi_1+\alpha_1, \ldots, \xi_m+\alpha_m\\ \xi_1, \ldots, \xi_m, \mu\end{matrix}\right) \times$$

$$G^{n+1,\,0}_{n,\,n+1}\left(qy\,\middle|\,\begin{matrix}\eta_1+\beta_1, \ldots, \eta_n+\beta_n\\ \eta_1, \ldots, \eta_n, \rho\end{matrix}\right) f(x, y)\,dx\,dy,\ R^*(p, q) > 0.$$

We shall denote this integral equation symbolically by

$$F(p, q) = G\left[f(x) : \begin{matrix}\alpha_m, \xi_m, \mu\\ \beta_n, \eta_n, \rho\end{matrix}\right].$$

When $\alpha_j = 0, j = 1, 2, \ldots, m-1$; $\beta_j = 0, j = 1, 2, \ldots n-1$ and

(*a*) $\alpha_m = 0, \mu = 0$; $\beta_n = 0, \rho = 0$, using $G^{1,\,0}_{0,\,1}(z/0) = e^{-z}$, (1·2) reduces to

$$(1\cdot3)\qquad F(p, q) = pq\int_0^\infty\int_0^\infty e^{-px-qy} f(x, y)\,dx\,dy,\ R(p, q) > 0.$$

*For the sake of brevity, the symbol $R(p, q) > 0$ has been used to denote $R(p) > 0$ $R(q) < 0$

which is denoted symbolically as

$$F(p, q) \doteqdot f(x, y)$$

and is known as Laplace transform of two variables[5].

(*b*) $\alpha_m = -M-k,\ \xi_m = M-k,\ \mu = -M-k\ ;\ \beta_n = -M_1-k_1,\ \eta_n = M_1-k_1,$ $\rho = -M_1-k_1$, (1·2) reduces to

$$(1\cdot4)\qquad F(p, q) = pq\int_0^\infty \int_0^\infty e^{-\frac{1}{2}px-\frac{1}{2}qy}(px)^{-k-\frac{1}{2}}(qy)^{-k_1-\frac{1}{2}} W_{k+\frac{1}{2},\,M}(px) \times W_{k_1+\frac{1}{2},\,M_1}(qy)\, f(x, y)\, dx\, dy,\ R(p, q)>0.$$

which shall be denoted by

$$F(p, q) \leftarrow \frac{k+\frac{1}{2},\ k_1+\frac{1}{2}}{M,\ M_1} f(x, y),$$

and is known as Meijer-Transform of two variables[3].

(*c*) $\xi_m = 2M,\ \alpha_m = \frac{1}{2}-k-M,\ \mu = 0\ ;\ \eta_n = 2M_1,\ \beta_n = \frac{1}{2}-k_1-M_1,\ \rho = 0,$ (1·2) reduces to

$$(1\cdot5)\qquad F(p, q) = pq\int_0^\infty \int_0^\infty e^{-\frac{1}{2}px-\frac{1}{2}qy}(px)^{M-\frac{1}{2}}(qy)^{M_1-\frac{1}{2}} W_{k,M}(px) \times W_{k_1,M_1}(qy)\, f(x,y)\, dx\, dy,\ R(p, q)>0.$$

which we shall call as Varma transform of two variables[9] and will denote symbolically by

$$F(p, q) \leftarrow \frac{k,\ k_1}{M,\ M_1} f(x, y).$$

2. The object of this paper is to obtain certain properties of Meijer-Laplace transform of two variables involving chain of Meijer-Laplace and Laplace transforms. These properties are generalisation of some of the known results in Laplace transform of two variables and yield as particular cases the results in other generalised Laplace transforms.

Theorem I.

If

$$F(p, q) \doteqdot f(x, y),$$

then

$$(2\cdot1)\qquad F\left(\frac{1}{p^c},\ \frac{1}{q^d}\right) = G\left[\frac{\prod_{j=1}^{m}\Gamma(\xi_j+\alpha_j+\nu)\ \prod_{j=1}^{n}(\eta_j+\beta_j+\delta)\ x^c y^d}{\prod_{j=1}^{m}\Gamma(\xi_j+\nu)\ \Gamma(\mu+\nu)\ \prod_{j=1}^{n}\Gamma(\eta_j+\delta)\ (\rho+\delta)}\right] \times$$

$$\times \int_0^\infty \int_0^\infty m_c F_{m c+c}\left\{\begin{matrix}\Delta^*(c,\ \xi_1+\alpha_1+\nu),\ \ldots,\ \Delta(c,\ \xi_m+\alpha_m+\nu) \\ \Delta(c,\ \xi_1+\nu),\ \ldots,\ \Delta(c,\ \xi_m+\nu),\ \Delta(c,\ \mu+\nu)\end{matrix} : -\frac{s\,x^c}{c^c}\right\} \times$$

*For the sake of brevity, symbol $\Delta(n, \alpha)$ has been used to denote the set of parameters

$$\frac{\alpha}{n},\ \frac{\alpha+1}{n},\ \ldots,\ \frac{\alpha+n-1}{n}.$$

$$\times {}_{nd}F_{nd+n}\left\{\begin{matrix}\Delta(d,\eta_1+\beta_1+\delta),\ldots,\Delta(d,\eta_n+\beta_n+\delta)\\ \Delta(d,\eta_1+\delta),\ldots,\Delta(d,\eta_n+\delta),\Delta(d,\rho+\delta)\end{matrix}:-\frac{t\,y^d}{d^d}\right\}\times$$

$$f(s,t)\,ds\,dt;\ \begin{matrix}\alpha_m,\ \xi_m,\ \mu\\ \beta_n,\ \eta_n,\ \rho\end{matrix}\Big].$$

where $\nu = c+1$, $\delta = d+1$; provided $R(\xi_j+\nu)>0, j=1,2,\ldots,m+1$, $\xi_{m+1}=\mu$; $R(\eta_j+\nu)>0, j=1,2,\ldots,n+1$, $\eta_{n+1}=\rho$, $f(x,y)$ is continuous for $x\geqslant\varepsilon_1>0$, $y\geqslant\varepsilon_2>0$ and the integrals converge absolutely.

Proof: Given $$(F(p,q)=pq)\int_0^\infty\int_0^\infty e^{-ps-qt}f(st)\,ds\,dt.$$

Replacing p and q by $\frac{1}{p^c}$ and $\frac{1}{q^d}$ respectively, we get

(2·2) $$F\left(\frac{1}{p^c},\frac{1}{q^d}\right)=\int_0^\infty\int_0^\infty p^{-c}q^{-d}e^{-\frac{s}{p^c}-\frac{t}{q^d}}f(s,t)\,ds\,dt.$$

We know

$$p^{-c}e^{-\frac{s}{p^c}}=\sum_{r=0}^{\infty}\frac{(-s)^r}{r!\,p^{cr+c}}\,G\Big[\frac{\prod_{j=1}^{m}\Gamma(\xi_j+\alpha_j+\nu)\,x^c}{\prod_{j=1}^{m}\Gamma(\xi_j+\nu)\,\Gamma(\mu+\nu)}\times$$

$${}_{mc}F_{mc+c}\left\{\begin{matrix}\Delta(c,\xi_1+\alpha_1+\nu),..,\Delta(c,\xi_m+\alpha_m+\nu)\\ \Delta(c,\xi_1+\nu),..,\Delta(c,\xi_m+\nu),\Delta(c,\mu+\nu)\end{matrix}:-\frac{s\,x^c}{c^c}\right\};\ \begin{matrix}\alpha_m,\ \xi_m,\ \mu\end{matrix}\Big],$$

where $\nu=c+1$; provided $R(\xi_j+\nu)>0, j=1,2,\ldots,m+1$, $\xi_{m+1}=\mu$ and

$$q^{-d}e^{-\frac{t}{q^d}}=G\Big[\frac{\prod_{j=1}^{n}\Gamma(\eta_j+\beta_j+\delta)\,y^d}{\prod_{j=1}^{n}\Gamma(\eta_j+\delta)\,\Gamma(\rho+\delta)}\times$$

$${}_{nd}F_{nd+d}\left\{\begin{matrix}\Delta(d,\eta_1+\beta_1+\delta),\ldots,\Delta(d,\eta_n+\beta_n+\delta)\\ \Delta(d,\eta_1+\delta),\ldots,\Delta(d,\eta_n+\delta),\Delta(d,\rho+\delta)\end{matrix}:-\frac{t\,y^d}{d^d}\right\}:\beta_n,\ \eta_n,\ \rho\Big],$$

where $\delta=d+1$; provided $R(\eta_j+\delta)>0, j=1,2,\ldots,n+1$, $\eta_{n+1}=\rho$.

By substituting these in (2·2), the result (2·1) follows, provided $R(\xi_j+\nu)>0$, $j=1,2,\ldots,m+1$, $\xi_{m+1}=\mu$; $R(\eta_j+\delta)>0$, $j=1,2,\ldots,n+1$, $\eta_{n+1}=\rho$ and the integrals converge absolutely.

Theorem II.

If $$F(p, q) \doteqdot f(x, y)$$

then

(2·3) $$\frac{F(\log p, \log q)}{\log p . \log q}$$

$$= G\left[\int_0^\infty \int_0^\infty \frac{\prod_{j=1}^{m} \Gamma(\xi_j+\alpha_j+s+1) \prod_{j=1}^{n} \Gamma(\eta_j+\beta_j+t+1)\, x^s y^t}{\prod_{j=1}^{m} \Gamma(\xi_j+s+1)\, \Gamma(\mu+s+1) \prod_{j=1}^{n} \Gamma(\eta_j+t+1)\, \Gamma(\rho+t+1)} f(s, t)\, ds\, dt\,;\right.$$

$$\left.\begin{matrix} \alpha_m, \xi_m, \mu \\ \beta_n, \eta_n, \rho \end{matrix}\right], \text{ provided } R(\xi_j+1) > 0, j = 1, 2, \ldots, m+1, \xi_{m+1} = \mu\,;$$

$R(\eta_j+1) > 0, j = 1, 2, \ldots, n+1, \eta_{n+1} = \rho\,; f(x, y)$ is continuous function for $x \geqslant \varepsilon_1 > 0, y \geqslant \varepsilon_2 > 0$ and the integrals are absolutely convergent.

Proof : Given

$$F(p, q) = pq \int_0^\infty \int_0^\infty e^{-ps-qt} f(s, t)\, ds\, dt.$$

Replacing p and q by $\log p$ and $\log q$ respectively, we get

(2·4) $$\frac{F(\log p, \log q)}{\log p . \log q} = \int_0^\infty \int_0^\infty p^{-s} q^{-t} f(s, t)\, ds\, dt.$$

Also, we know

$$p^{-s} = G\left[\frac{\prod_{j=1}^{m} \Gamma(\xi_j+\alpha_j+s+1)\, x^s}{\prod_{j=1}^{m} \Gamma(\xi_j+s+1)\, \Gamma(\mu+s+1)}; \alpha_m, \xi_m, \mu\right],$$

and

$$q^{-t} = G\left[\frac{\prod_{j=1}^{n} \Gamma(\eta_j+\beta_j+t+1)\, y^t}{\prod_{j=1}^{n} \Gamma(\eta_j+t+1)\, \Gamma(\rho+t+1)}; \beta_n, \eta_n, \rho\right].$$

substituting these in (2·4), we get the result (2·3), provided $R(\xi_j+1) > 0$, $j = 1, 2, \ldots, m+1$, $\xi_{m+1} = \mu$; $R(\eta_j+1) > 0, j = 1, 2, \ldots, n+1$, $\eta_{n+1}{}^{\mu} =$ and the integrals are absolutely convergent.

Theorem III.

If
$$F(p, q) = G\left[f(x, y); \begin{matrix} \alpha_m, \xi_m, \mu \\ \beta_n, \eta_n, \rho \end{matrix}\right],$$
and
$$p^{-r_1} q^{-r_2} f(p, q) \doteqdot \phi(x, y),$$
then

$$F(p, q) = \frac{\prod_{j=1}^{m} \Gamma(\xi_j + r_1 + 2)\, \Gamma(\mu + r_1 + 2) \prod_{j=1}^{n} \Gamma(\eta_j + r_2 + 2)\, \Gamma(\rho + r_2 + 2)}{\prod_{j=1}^{m} \Gamma(\xi_j + \alpha_j + r_1 + 2) \prod_{j=1}^{n} \Gamma(\eta_j + \beta_j + r_2 + 2)\, p^{r_1+1} q^{r_2+2}} \times$$

$$(2\cdot5) \qquad \times \int_0^\infty \int_0^\infty {}_{m+1}F_m\left[\begin{matrix} \xi_1 + r_1 + 2, \ldots, \xi_m + r_1 + 2, \mu + r_1 + 2 \\ \xi_1 + \alpha_1 + r_1 + 2, \ldots, \xi_m + \alpha_m + r_1 + 2 \end{matrix}; -\frac{x}{p}\right] \times$$

$${}_{n+1}F_n\left[\begin{matrix} \eta_1 + r_2 + 2, \ldots, \eta_n + r_2 + 2, \rho + r_2 + 2 \\ \eta_1 + \beta_1 + r_2 + 2, \ldots, \eta_n + \beta_n + r_2 + 2 \end{matrix}; -\frac{y}{p}\right] \phi(x, y)\, dx\, dy$$

provided $R(\xi_j + r_1 + 2) > 0, j = 1, 2, \ldots, m+1$, $\xi_{m+1} = \mu$; $R(\eta_j + r_2 + 2) > 0$, $j = 1, 2, \ldots, n+1$, $\eta_{n+1} = \rho$; $\phi(x, y)$ is continuous function for $x \geqslant \varepsilon_1 > 0$, $y \geqslant \varepsilon_2 > 0$, and the integrals are absolutely convergent.

Proof: Given

$$(2\cdot6) \qquad F(p, q) = pq \int_0^\infty \int_0^\infty G^{m+1, 0}_{m, m+1}\left(px \,\middle|\, \begin{matrix} \xi_1 + \alpha_1, \ldots, \xi_m + \alpha_m \\ \xi_1, \ldots, \xi_m, \mu \end{matrix}\right) \times$$

$$G^{n+1, 0}_{n, n+1}\left(qy \,\middle|\, \begin{matrix} \eta_1 + \beta_1, \ldots, \eta_n + \beta_n \\ \eta_1, \ldots, \eta_n, \rho \end{matrix}\right) f(s, t)\, ds\, dt,$$

and

$$(2\cdot7) \qquad f(s, t) = s^{r_1+1} t^{r_2+1} \int_0^\infty \int_0^\infty e^{-sx-ty} \phi(x, y)\, dx\, dy.$$

Substituting the value of $f(s,t)$ from (2·7) in (2·6) and changing the order of integration, which is permissible under the conditions stated in the theorem by de la Vallee Poussin's theorem [4, p. 504],

we get

$$(2\,8) \quad F(p, q) = \int_0^\infty \int_0^\infty \phi(x, y) \left[pq \int_0^\infty \int_0^\infty G^{m+1, 0}_{m, m+1}\left(ps \,\middle|\, \begin{matrix} \xi_1 + \alpha_1, \ldots, \xi_m + \alpha_m \\ \xi_1, \ldots, \xi_m, \mu \end{matrix}\right) \times \right.$$

$$\left. G^{n+1, 0}_{n, n+1}\left(qy \,\middle|\, \begin{matrix} \eta_1 + \beta_1, \ldots, \eta_n + \beta_n \\ \eta_1, \ldots, \eta_n, \rho \end{matrix}\right) s^{r_1+1} t^{r_2+1} e^{-sx-ty}\, ds\, dt \right] dx\, dy,$$

evaluating the inner double integral with the help of a known result [6, p. 419, (5)] and then expressing Meijer's G-function as Hypergeometric function, we get (2·5).

Theorem IV.

If $$F(p, q) = G\left[f(x, y) ; \begin{matrix} a_m, \xi_m, \mu \\ \beta_n, \eta_n, \rho \end{matrix} \right],$$

and $$\sqrt{pq}\, f\left(\frac{1}{p}, \frac{1}{q}\right) = G\left[g(x, y) ; \begin{matrix} \alpha_m, \xi_m, \mu \\ \beta_n, \eta_n, \rho \end{matrix} \right],$$

then

$$(2\cdot9) \qquad F(p^2, q^2) = G\left[\frac{\pi x y}{2^{2(1+\mu+\rho-\sum_{j=1}^{m} \alpha_j - \sum_{j=1}^{n} \beta_i)}} g\, \frac{x^2}{4}, \frac{y^2}{4} : \begin{matrix} 2a_m, 2\xi_m, 2\mu \\ \beta_n, 2\eta_n, 2\rho \end{matrix} \right]$$

provided the integrals are absolutely convergent.

Proof: We have

$$F(p, q) = pq \int_0^\infty \int_0^\infty G^{\,m+1,\,0}_{m,\,m+1}\left(px \,\middle|\, \begin{matrix} \xi_1+\alpha_1, \ldots, \xi_m+\alpha_m \\ \xi_1, \ldots, \xi_m, \mu \end{matrix} \right) \times$$

$$G^{\,n+1,\,0}_{n,\,n+1}\left(qy \,\middle|\, \begin{matrix} \eta_1+\beta_1, \ldots, \eta_n+\beta_n \\ \eta_1, \ldots, \eta_n, \rho \end{matrix} \right) f(x, y)\, dx\, dy,$$

$$(2\cdot10) \qquad = pq \int_0^\infty \int_0^\infty G^{\,m+1,\,0}_{m,\,m+1}\left(\frac{p}{x} \,\middle|\, \begin{matrix} \xi_1+\alpha_1, \ldots, \xi_m+\alpha_m \\ \xi_1, \ldots, \xi_m, \mu \end{matrix} \right) \times$$

$$G^{\,n+1,\,0}_{n,\,n+1}\left(\frac{q}{y} \,\middle|\, \begin{matrix} \eta_1+\beta_1, \ldots, \eta_n+\beta_n \\ \eta_1, \ldots, \eta_n, \rho \end{matrix} \right) \sqrt{xy}\, f\left(\frac{1}{x}, \frac{1}{y}\right) \frac{dx\, dy}{x^2\sqrt{x}.\, y^2\sqrt{y}}.$$

Since

$$(2\cdot11) \qquad \sqrt{pq}\, f\left(\frac{1}{p}, \frac{1}{q}\right) = G\left[g(x, y) ; \begin{matrix} a_m, \xi_m, \mu \\ \beta_n, \eta_n, \rho \end{matrix} \right]$$

Substituting for $\sqrt{xy}\ f\left(\frac{1}{x}, \frac{1}{y}\right)$ from (2·11) in (2·10), inter-changing the order of integration which is permissible due to absolute convergency of the integrals involved, after some simplification using [6, p. 422, (14)], [7, p. 209, (10)] and simplifying again we obtain the result (2·9).

Particular Cases

Taking $\alpha_j = 0, j = 1, 2, \ldots, m, \mu = 0$; $\beta_j = 0, j = 1, 2, \ldots, n, \rho = 0$ and specializing the parameters suitably, results (2·1), (2·3), (2·5) and (2·9) yield as particular cases, the results due to Bose [2, p. 175–177].

Setting $\alpha_j = 0, j = 1, 2, \ldots, m-1, \alpha_m = -M-k, \xi_m = M-k, \mu = -M-k$; $\beta_j = 0, j = 1, 2, \ldots, n-1, \beta_n = -M_1-k_1, \eta_n = M_1-k_1, \rho = -M_1-k_1$, and giving suitable values to the parameters in (2·1), (2·3) and (2·5) we get as particular cases the results of Bose and Mehra [3, p. 53, (17); p. 52, (14); p. 50, (10)].

Putting $\alpha_j = 0, j = 1, 2, \ldots, m-1$, $\alpha_m = -M-k$, $\xi_m = M-k$, $\mu = -M-k$; $\beta_j = 0, j = 1, 2, \ldots, n-1$, $\beta_n = -M_1-k_1$, $\eta_n = M_1-k_1$, $\rho = -M_1-k_1$ in (2·9), we get the corresponding result in Meijer transform of two variable.

If $\alpha_j = 0, j = 1, 2, \ldots, m-1$, $\alpha_m = \frac{1}{2}-k-M$, $\xi_m = 2M$, $\mu = 0$; $\beta_j = 0$, $j = 1, 2, \ldots, n-1$, $\beta_n = \frac{1}{2}-k_1-M_1$, $\eta_n = 2M_1$, $\rho = 0$, in (2·1), (2·3), (2·5) and (2·9), corresponding results in Varma transform of two variable are obtained.

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References

1. Bhise, V. M. Inversion formulae for a generalised Laplace integral. *Jour. Vikram Univ.*, 3(3), 57–63, 1959.
2. Bose, S. K. On Laplace transform of two variables. *Bull. Cal. Math. Soc.*, **41**, 173–178, 1949.
3. Bose, S. K. and Mehra, A. N. On Meijer transform of two variables. *Ganita*, **9**(1), 1958.
4. Bromwich, T. J. I'A. An introduction to the theory of Infinite Series. *Macmillan, New York*, 1955.
5. Ditkin, V. A. and Prudnikov. Operational Calculus in two variable. *Pergamon Press*, 1962.
6. Erdelyi, A. Tables of Integral transform Vol. II, *McGraw Hill*, 1954.
7. Erdelyi, A. Higher Transcendental Functions. Vol. I, *McGraw Hill*, 1953.
8. Jain, N. C. Inversion formulae for Meijer-Laplace transform of two variables. *Communicated.*
9. Mukherjee, S. N. Some inversion formulae for the generalised Laplace transform of two variables. *Vijnana Anusandhan Patrika*, **5**, 49–55, 1962.

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On the spiral structure as an explosive phenomenon

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Abstract

In the present paper we have considered spiral arm formation to be due to the explosion of the galactic nucleus. The Magellanic Clouds have been proposed to be an off-shoot of the Galaxy, having evolved because of explosions of the galactic nucleus, in its early evolutionary stage.

Assuming that the ellipsoidal velocity distribution holds we have investigated the condition necessary for the formation of the spiral arm.

The velocity of ejection (V_{ej}), as a function of the distance of the point of ejection (R_{ej}) and the final attainable distance (R_2), has been determined. For $R_2 = 6$ kpc the variation of transversal velocity, with the radial velocity and the distance of the point of ejection, has also been studied.

Introduction

In a previous paper[1] we considered the formation of the spiral structure, of the Galaxy, to be due to the spilling of matter in the galactic plane, as a result of instability produced by the rapid rotation of a central mass. The galactic force laws used were those given by Oort and Bottlinger. The condition for the formation of the spiral arms was also deduced from on expression for potential derived by Parenago.

In the present investigation, however, we have considered the explosion of the galactic nucleus as an alternative mechanism for the spiral structure.

Explosions of the galactic nucleus are considered to be a very likely phenomenon in the galaxy. According to Ishida[2] 'all normal galaxies must have experienced enormous explosions of the nucleus 1–10 times in 10^9 years'. Seyfert and radio galaxies are believed to be caused by the explosion of the nucleus. Lynds and Sandage[3] have reported evidences of an explosion in the central region of M82. This has been regarded as the most direct evidence of an explosion of the nucleus of a galaxy. The Burbidges[4] have arrived at the same conclusion, of an explosive activity of M82, from a study of the velocity field in that galaxy. Aizu *et al*[5] have proposed a new theory, called the pile theory, to explain the mechanism of the explosion. Besides extra-galactic objects, evidences of explosive activities for the Galaxy is not lacking. The flowin of gas at the solar neighbourhood, from the halo, has been considered by Ishida[2] to be the result of the explosion $\gtrsim 5 \times 10^7$ years ago. The presence of the large galactic halo has been suggested by him to be due to the explosion of the nucleus 10^8–10^9 years ago. The 3-kpc-arm (which is moving towards us with a radial velocity of 50 km/sec) has been considered to be expanding due to an explosion of the galactic nucleus $\sim 2 \times 10^7$ years ago.

The explosion of the galactic nucleus seems to suggest a new view on the origin of the Magellanic Clouds. They may be considered to be a 'product' of the Galaxy, hurled at their present distance due to explosions of the galactic nucleus, in its early evolutionary stage. It is felt, at this stage, decisive proofs can not be furnished. Later works, however, are expected to support or criticize this view.

Formulation of problem and solution of the basic equation :

Assuming that the ellipsoidal velocity distribution holds, for the Galaxy, then the galactic force law $F(r)$ can be written as[6]

$$F(r) = \frac{k_1^2 r}{(1 + k_2 r^2)^2} \tag{1}$$

where the values of k_1 and k_2 are given by

$$k_1 = 7{\cdot}2\ km/\text{sec}/kpc = 2{\cdot}34 \times 10^{-15}\ \text{sec}^{-1}$$

$$k_2 = {\cdot}0237\ kpc^{-2} = 2{\cdot}49 \times 10^{-45}\ cm^{-2} \tag{2}$$

These values are slightly uncertain and can be improved upon by using the following recent estimates of galactic parameters, viz :

$$R_\circ = 10\ kpc,\ A = 15\ km/\text{sec per } kpc, \text{ and } \omega_\circ = 25\ km/\text{sec}/kpc$$

The equation of motion, in the equatorial plane, when affect due to viscous and magnetic forces are neglected, is given by

$$\frac{d^2u}{d\theta^2} + u = \frac{k_1^2 u}{h^2 (u^2 + k_2)^2} \tag{3}$$

Multiplying by $2\frac{du}{d\theta}$, and interegrating equation (3)

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{k^{12}}{h^2} \int \frac{2u}{(u^2 + k_2)^2}\, du \tag{4}$$

or

$$v^2 = h^2 \left\{ \left(\frac{du}{d\theta}\right) + u^2 \right\} = C - \frac{k_1^2}{u^2 + k_2} \tag{5}$$

where v is the velocity at any point $r = \frac{1}{u}$ of the path. Now $v = V_{ej}$ at $r = R_{ej}$ Hence the arbitrary constant is given by

$$C = V_{ej}^2 + \frac{k_1^2 R_{ej}^2}{1 + k_2 R_{ej}^2} \tag{6}$$

or

$$v^2 = V_{ej}^2 + k_1^2 \left(\frac{R_{ej}^2}{1 + k_2 R_{ej}^2} - \frac{r^2}{1 + k_2 r^2} \right) \tag{7}$$

The expression for the finite length of the arm, R_2, will be expressed as

$$R_2 = \left\{ \frac{V_{ej}^2 + k_2 V_{ej}^2 R_{ej}^2 + k_1^2 R_{ej}^2}{k_1^2 = k_2 V_{ej}^2 - k_2^2 V_{ej}^2 R_{ej}^2} \right\}^{1/2} \tag{8}$$

The condition for the outward flow of matter will be

$$V_{ej}^2 + \frac{k_1^2 R_{ej}^2}{1 + k_2 R_{ej}^2} > \frac{k_1^2 r^2}{1 + k_2 r^2} \tag{9}$$

for values of r greater than R_{ej}.

The expression for the velocity of ejection with the help of equation (7) can be written as

$$V_{ej} = \left\{ \frac{k_1^2 (R_2^2 - R_{ej}^2)}{(1 + k_2 R_2^2)(1 + k_2 R_{ej}^2)} \right\}^{1/2} \tag{10}$$

and the velocity (V_{inf}), to detach the particle from the galactic system, from a given distance of ejection (R_{ej}), can be written as

$$V_{inf} = \frac{k_1}{(k_2 + k_2^2 R_{ej}^2)^{1/2}} \tag{11}$$

In Table 1 values of V_{ej} (km/sec) are given corresponding to the values of R_2 (kpc) and R_{ej} (kpc).

TABLE 1

R_2 (kpc)→ / R_{ej} ↓ (kpc)	6	8	10	12	14	∞
·5	316	362	392	411	424	468
1·0	310	357	387	406	419	463
1·5	300	348	379	399	412	457
2·0	287	337	368	389	402	448
2·5	270	323	355	376	391	438
3·0	250	306	341	363	377	426

If the radial velocity, V_r, is known the transversal velocity, V_t, is easily seen to be given by

$$V_t^2 = \frac{k_1^2 (R_2^2 - R_{ej}^2)}{(1 + k_2 R_2^2)(1 + k_2 R_{ej}^2)} - V_r^2 \tag{12}$$

In Table 2, corresponding to $R_2 = 6\ kpc$ we have calculated the transversal velocity as a function of the radial velocity and the distance of the point of ejection.

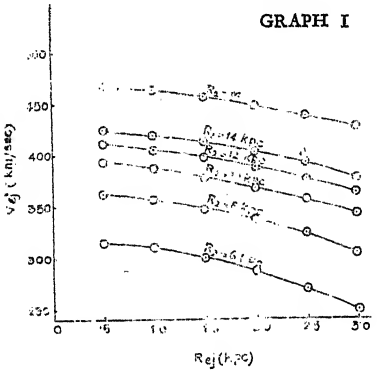
TABLE 2

V_{rad} (km/sec)→ / R_{ej} ↓ (kpc)	20	40	50	60	80	100
·5	315	313	312	310	306	300
1·0	309	307	306	304	299	293
1·5	299	298	296	294	289	283
2·0	286	280	282	280	275	269
2·5	269	267	265	263	258	251
3·0	249	247	245	243	237	229

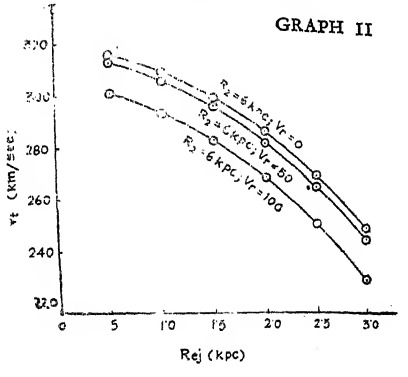
In Graph I, V_{ej} is studied as a function of R_2 and R_{ej}.

In Graph II, for $R_2 = 6$ *kpc*, V_t is studied as a function of V_r and R_{ej}.

GRAPH I



GRAPH II



For other values of R_2, the transversal velocity, as a function of V_r and R_{ej}, will exhibit the same general characteristics.

In another communication it is proposed to discuss the spiral arm formation, as an explosive phenomenon, in the light of the galactic force law given by Oort and Bottlinger. Since the galactic phenomenon and in particular the force law is not completely understood. Comparatively study of these results are expected to be interesting as well as useful.

Acknowledgments

It is a pleasure to express my sincere thanks to Prof. A. C. Banerji, Ex-Vice-Chancellor and Emeritus Professor of Mathematics, University of Allahabad, Allahabad, for many stimulating and helpful discussions during this investigation. My thanks are also to Prof. H. F. Weaver, Radio Astronomy Laboratory, University of California, U. S. A., for suggesting the force law. In the end I thank the Council of Scientific and Industrial Research for the award of the Research grant.

References

1. Banerji, A. C. and Gurtu, S. K. (in press).
2. Ishida, K. *Supp. Prog. Theo. Phy.*, 31, 116-130, 1964.
3. Lynds, G. R. and Sandage, A. R. *Ap. J.*, **137**, 1008-1021, 1963.
4. Burbidge, E. M. *et al.* *Ap. J.* **149** (3), 942-1003, Oct., 1964.
5. Aizu, K. *et al.* *Supp. Prog. Theo. Phy.*, 31, 35-61, 1964.
6. Weaver, H. F. (Private communication).
7. Ogorodnikov, K. F. Dynamics of Stellar systems. *Pergamon Press, The Macmillan Co., New York*, 192, 1965.

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On the spiral structure as an explosive phenomenon—II

By

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Abstract

Assuming spiral arm formation to be due to the explosion of the galactic nucleus we have determined the necessary condition for its formation using the expression for the galactic force law given by Oort, Bottlinger and Schmidt.

Numerical calculations have been carried out for the galactic force law due to Schmidt. The velocity of ejection (V_{ej}), as a function of the point of ejection (R_{ej}) and the final attainable distance (R_2), has been determined.

The numerical results are found to be in good agreement with that derived from a previous investigation.

Introduction

In a previous paper[1] we considered the formation of the spiral arms, of the Galaxy, to be due to matter being hurled in the galactic disc as a result of instability produced by the rapid rotation of a central mass. The galactic force laws used were those given by Oort, Bottlinger and Parenago. In another communication[2], however, we suggested the spiral arm formation to be due to the explosion of the galactic nucleus. The condition for the ejection of matter and other numerical results were deduced by assuming an ellipsoidal velocity distribution to hold for the Galaxy.

Ishida[3] considers explosions of the galactic nucleus a very likely phenomenon in the galaxy, a fact corroborated by observational evidences of several investigators[4-6]. Consequently, in the present paper, we will attribute the expanding activity of the 3-kpc-arm, which is a full fledged arm according to Oort and others[7], in terms of the explosion of the galactic nucleus.

1. Oort's Law :

The galactic force law, proposed by Oort, as a function of the distance from centre, is[1]

$$P(0) = \frac{C_1}{r_2} + C_2 r \tag{1}$$

where $C_1 = 4{\cdot}59 \times 10^{12}$ and $C_2 = 2{\cdot}7 \times 10^{-16}$

The equation of motion, in the equatorial plane, under the influence of the above central force law, and when effect due to viscous, magnetic forces etc. are neglected, is given by

$$\frac{d^2u}{d\theta^2} + u = \frac{P(0)}{h^2 u^2} = \frac{1}{h^2}\left(C_1 + \frac{C_2}{u^3} \right) \tag{2}$$

Multiplying by $2\frac{du}{d\theta}$, and integrating equation (2)

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{2}{h^2}\int \left(C_1 + \frac{C_2}{u^3}\right) du \tag{3}$$

or
$$v^2 = C + 2\left(C_1 u - \frac{C_2}{2u^2}\right) \tag{4}$$

where v is the velocity at any point $r = 1/u$ of the path. Now $v = V_{ej}$ at $r = R_{ej}$ hence the arbitrary constant is given by

$$C = V_{ej}^2 - 2\left(\frac{C_1}{R_{ej}} - \frac{C_1}{2} R_{ej}^2\right) \tag{5}$$

The velocity at any point of the path will be

$$v^2 = V_{ej}^2 - \frac{2C_1}{r R_{ej}} (r - R_{ej}) - C^2 (r^2 - R_{ej}^2) \tag{6}$$

and the expression for the finite length of the arm, R_2, will be given by the cubic

$$(C_2 R_{ej}) R_2^3 + (2C_1 - C_2 R_{ej}^3 - V_{ej}^2 R_{ej}) R_2 - 2C_1 R_{ej} = 0. \tag{7}$$

The condition for the outward flow of matter will be

$$V_{ej}^2 > 2C_1 \frac{r - R_{ej}}{r R_{ej}} + C_2 (r^2 - R_{ej}^2) \tag{8}$$

For values of r greater than R_{ej}
The velocity of ejection, as a function of R_2 and R_{ej}, will be expressed as

$$V_{ej}^2 = \frac{2C_1}{R_2 R_{ej}} (R_2 - R_{ej}) + C_2(R_2^2 - R_{ej}^2) \tag{9}$$

2. Bottlinger's law :

Bottlinger proposed the formula, for the galactic force law, as[1]

$$P(B_1) = \frac{ar}{1 + b r^3} \tag{10}$$

We find the velocity, at any point $r = 1/u$, will be given by

$$v^2 = \frac{2a}{3g^2}\left[\log (u + g) - \tfrac{1}{2}\log (u^2 - gu + g^2) + \sqrt{3}\tan^{-1}\frac{2u - g}{\sqrt{3}g}\right] \tag{11}$$

where $b = g^3$ and

$$D = V_{ej}^2 - \frac{2a}{3g^2}\left[\log (1 + g R_{ej}) - \tfrac{1}{2}\log (1 - g R_{ej} + g^2 R_{ej}^2) + \sqrt{3}\tan^{-1}\frac{2 - g R_{ej}}{\sqrt{3} g R_{ej}}\right] \tag{12}$$

and the expression for the finite length of the arm, R_2, by the equation.

$$V_{ej}^2 - \frac{2a}{3g^2}\left[\log\frac{1+gR_{ej}}{1+gR_2} - \tfrac{1}{2}\log\frac{1-gR_{ej}+g^2R_{ej}^2}{1-gR_2+g^2R_2^2} - \sqrt{3}\left(\tan^{-1}\frac{2-gR_{ej}}{\sqrt{3}gR_{ej}} - \tan^{-1}\frac{2-gR_2}{\sqrt{3}gR_2}\right)\right] = 0 \quad (13)$$

It can be easily seen that equation (13) will give the velocity of ejection as a function of R_2 and R_{ej}. The required condition for the outward flow of matter will be

$$V_{ej}^2 > \frac{2a}{3g^2}\left[\log\frac{1+gR_{ej}}{1+gr} - \tfrac{1}{2}\log\frac{1-gR_{ej}+g^2R_{ej}^2}{1-gr+g^2r^2} + \sqrt{3}\left(\tan^{-1}\frac{2-gR_{ej}}{\sqrt{3}gR_{ej}} - \tan^{-1}\frac{2-gr}{\sqrt{3}gr}\right)\right] \quad (14)$$

for values of r greater than R_{ej}.

2. Schmidt's Law :

The galactic force law, proposed by Schmidt[8], can be written as

$$P(S) = Pr + Q \quad (15)$$

where the values of P and Q can be expressed in terms of the galactic parameter A, B and R_o. Their numerical values can be written as[8]

$$P = -1362\ km^2/\sec^2\ kpc^2$$
$$Q = 16885\ km^2/\sec^2\ kpc \quad (16)$$

To determine P and Q Schmidt adopted the following numerical values for the galactic parameters :

$$A = 19{\cdot}5\ km/\text{sec.}\ kpc,\ \omega = 26{\cdot}4\ km/\text{sec.}\ kpc \text{ and } R_0 = 8{\cdot}2\ kpc.$$

Comparison of the above force law with the force actually observed for the galactic system reveals the agreement is quite good : between $r = 2{\cdot}5$ and $8{\cdot}2\ kpc$ the deviations never exceed 6%. It will be worthwhile to extend our investigation to this force law, as well.

The velocity at any point of the path, distant r from the centre, will easily seen to be given by

$$V^2 = V_{ej}^2 + (PR_{ej}^2 + 2QR_{ej}) - (Pr^2 + 2Qr) \quad (17)$$

and the expression for the finite length of the arm, R_2 will be given by the quadratic.

$$PR_2^2 + 2QR_2 - (V_{ej}^2 + PR_{ej}^2 + 2QR_{ej}) = 0 \quad (18)$$

In order that R_2 may have roots

$$Q^2 + P(V_{ej}^2 + PR_{ej}^2 + 2QR_{ej}) > 0 \quad (19)$$

The required condition, for the outward flow of matter, is

$$V_{ej}^2 + PR_{ej}^2 + 2QR_{ej} > Pr^2 + 2Qr \quad (20)$$

for values of r greater than R_{ej}

The velocity (V_{ej}) with which a particle be ejected from a given distance (R_{ej}), so that it may reach a particular distance (R_2), from the galactic centre, will be given by

$$V_{ej} = [(R_2 - R_{ej}) \{P (R_2 + R_{ej}) + 2 Q\}]^{1/2} \quad (21)$$

In Table 1 values of V_{ej} (*km*/sec) are given corresponding to the values of R_2 (*kpc*) and R_{ej} (*kpc*).

TABLE 1

$R_2(kpc)\rightarrow$ / $R_{ej}(kpc)\downarrow$	6	8	10	12
·5	370	408	431	439
1·0	348	388	411	420
1·5	326	368	392	402
2·0	303	348	374	383
2·5	279	327	355	365
3·0	254	307	335	347

Comparison of the above numerical results, with that obtained earlier[2], reveals, they are in good agreement. It may be observed, in the above mentioned two investigations, we have adopted different values for A, B and R_o. It would be better if, in the above investigations, recent estimates of the galactic parameters, as recommended by the 12th General Assembly of IAU held at Hamburg in 1964, are used as follows :

$$A = 15 \; km/\text{sec}. \; kpc, \; \omega = 25 \; km/\text{sec}. \; kpc \text{ and } R_o = 10 \; kpc.$$

So far, no major objections to these values have been raised. It is hoped, however, these new values will not very seriously impair the validity of our conclusion.

Acknowledgements

I place on record my indebtedness to Prof. A. C. Banerji, Ex-Vice-Chancellor and Emeritus Professor of Mathematics, University of Allahabad, Allahabad, for his kind guidance and unfailing encouragement in the preparation of this paper. I also thank the Council of Scientific and Industrial Research for the award of the research grant.

References

1. Banerji, A. C. and Gurtu, S. K. (In press).
2. Banerji, A. C. and Gurtu, S. K. (under communication)
3. Ishida, K. *Supp. Prog Theo. Phy.*, 31, 116–130, 1964.
4. Lynds, C. R. and Sandage, A. R. *Ap. J.*, **137**, 1008–1021, 1963.
5. Burbidge, E. M. *et al.* *Ap. J.*, **140**,(3), 942–1008, 1966.
6. Aizu, K. *et al.* *Supp. Prog. Theo. Phy.*, 31, 35–61, 1964.
7. Oort, J. H. *et al.* *Month. Not. Roy. Astron. Soc.*, **118**, 379–389, 1958.
8. Schmidt, M. *B. A. N.*, **13**, 468, 15–41, 1956.

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On some integral equations involving Jacobi polynomials

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Abstract

The solutions of the integral equations involving Jacobi polynomial are obtained by making use of Weber-Schafheitlin integral.

1. Recently Srivastava[1] has obtained the inversion formulae for certain integral transform involving Jacobi polynomials as their kernels, by making use of the result of Tranter [5, (8·3) (8·5), pp. 112–113] and two other results. All those three results are the special cases of Weber-Schafheitlin integral [2 p. 35].

In this paper we derive three results from Weber-Schafheitlin integral which are different from those used by[1]. Making use of these results we discuss the solutions of the integral equations involving Jacobi polynomials. The method adopted is the same as that of[1]. It will be observed that the integral equations, their solutions and conditions of validity are different in many respects than those of[1].

The result [5, pp. 112, 113] derived by Tranter, has been used by Tranter[6,7,8] and by Srivastava[1,9]. We could derive a result which is numbered (2·4) in this paper. This result is analogous to that of Tranter [5, pp. 112–113]. We have used this result elsewhere also.

2. The following results have been used :

If $Re\,(\mu + \nu - \lambda + 1) > 0$, $Re\,\lambda > -1$, a and b are real, then

$$(2\cdot1)\qquad \int_0^\infty J_\mu(at)\,J_\nu(bt)\,t^{-\lambda}\,dt =$$

$$\frac{a^\mu\,\Gamma\left(\frac{\mu+\nu-\lambda+1}{2}\right)}{2^\lambda\,b^{\mu-\lambda+1}\,\Gamma\left(\frac{-\mu+\nu+\lambda+1}{2}\right)\Gamma(\mu+1)}\;{}_2F_1\left(\frac{\mu+\nu-\lambda+1}{2},\frac{\mu-\nu-\lambda+1}{2};\ \mu+1;\frac{a^2}{b^2}\right),\qquad 0<a<b,$$

$$=\frac{b^\nu\,\Gamma\left(\frac{\mu+\nu-\lambda+1}{2}\right)}{2^\lambda\,a^{\nu-\lambda+1}\,\Gamma\left(\frac{\mu-\nu+\lambda+1}{2}\right)\Gamma(\nu+1)}\;{}_2F_1\left(\frac{\mu+\nu-\lambda+1}{2},\frac{-\mu+\nu-\lambda+1}{2};\ \nu+1;\frac{b^2}{a^2}\right),\qquad 0<b<a\ \text{[2, p. 35]}.$$

(2·2) $${}_2F_1(a, b ; c ; z) = (1 - z)^{c-a-b}\, {}_2F_1(c - a, c - b ; c ; z) \quad [3, \text{p. } 60].$$

(2·3) $$F_m(\alpha, \gamma, x) = {}_2F_1(-m, \alpha + m ; \gamma ; x)$$

where $F_m(\alpha, \gamma, x)$ is the Jacobi polynomial [2, p. 83]

By using the relations (2·1), (2·2) and (2·3), the following relations can easily be derived :

(2·4) If m is a non-negative integer such that $\nu > m - k + 1$ and $k < 2$, then

$$\int_0^\infty t^{k-1} J_{\nu-2m+k-2}(t)\, J_\nu(rt)\, dt = 0, \qquad 0 < r < 1,$$

$$= \frac{\Gamma(\nu - m + k - 1)\, 2^{k-1} r^{2m-\nu} (r^2 - 1)^{-k+1}}{\Gamma(m - k + 2)\, \Gamma(\nu - 2m + k - 1)} F_m\left(\nu - 2m, \nu - 2m + k - 1, \frac{1}{r^2}\right), \quad r > 1.$$

(2·5) If m is a non-negative integer such that $\nu > m$ and $k > 1$, then

$$\int_0^\infty t^{-k+2} J_{\nu-2m+k-2}(t)\, J_{\nu-1}(rt)\, dt$$

$$= \frac{\Gamma(\nu - m)\, r^{\nu-1}\, {}_2F_1(\nu - m, m - k + 2 ; \nu ; r^2)}{2^{k-2}\, \Gamma(-m + k - 1)\, \Gamma(\nu)}, \quad 0 < r < 1$$

$$= \frac{\Gamma(\nu - m)\, {}_2F_1\left(\nu - m, -m + 1 ; \nu - 2m + k - 1 ; \frac{1}{r^2}\right)}{2^{k-2}\, r^{\nu-2m+1}\, \Gamma(m)\, \Gamma(\nu - 2m + k - 1)}, \quad r > 1$$

(2·6) If m is a non-negative integer such that $\nu > m$ and $k > 1$, then

$$\int_0^\infty t^{-k+2} J_{\nu-2m+k-3}(t)\, J_\nu(rt)\, dt$$

$$= \frac{r^\nu\, \Gamma(\nu - m)\, {}_2F_1(\nu - m, m - k + 3 ; \nu + 1 ; r^2)}{2^{k-2}\, \Gamma(-m + k - 2)\, \Gamma(\nu + 1)}, \quad 0 < r < 1$$

$$= \frac{\Gamma(\nu - m)\, {}_2F_1\left(\nu - m, -m ; \nu - 2m + k - 2 ; \frac{1}{r^2}\right)}{2^{k-2}\, r^{\nu-2m}\, \Gamma(m + 1)\, \Gamma(\nu - 2m + k - 2)}, \quad r > 1$$

3. In this paper, we obtain the following two theorems :

Theorem I. Let

(3·1) $$\phi(x) = \int_0^x u^{\nu-2m+k-1} (x^2 - u^2)^{-k+1} F_m\left(\nu - 2m, \nu - 2m + k - 1, \frac{u^2}{x^2}\right) \psi(u)\, du$$

then

(3·2) $$\psi(x) = \frac{2\Gamma(\nu - m)\, \Gamma(\nu - m + k - 1)\, x^{k-\nu-2}}{\Gamma(-m + k - 1)\, \Gamma(\nu)\, \Gamma(\nu - 2m + k - 1)\, \Gamma(m - k + 2)} \times$$

$$\int_0^x {}_2F_1\left(\nu-m,\, m-k+2\,;\,\nu\,;\,\frac{u^2}{x^2}\right) d\left[u^{2m}\,\phi(u)\right]$$

$$+\frac{2\Gamma(\nu-m)\,\Gamma(\nu-m+k-1)\,x^{\nu-2m+k-2}}{\Gamma(m)\,\Gamma(\nu-2m+k-1)\,\Gamma(m-k+2)\,\Gamma(\nu-2m+k-1)}$$

$$\int_x^\infty u^{2m-2\nu}\,{}_2F_1\left(\nu-m\,,\,-m+1\,;\,\nu-2m+k-1\,;\,\frac{x^2}{u^2}\right) d[u^{2m}\,\phi(u)]$$

provided

(*i*) m is a non-negative integer such that $\nu > m$ and $1 < k < 2$

(*ii*) $\int_0^\infty t\,|\,\psi(t)\,|\,dt$ and $\int_0^\infty t^{1-\nu}\,\left|\frac{d}{dt}\{t^{2m}\,\phi(t)\}\right|\,dt$

are convergent,

(*iii*) $\frac{d}{dt}\{t^{2m}\,\phi(t)\}$ is continuous.

Theorem II. Let

$$(3\cdot3)\quad \phi(x)=\int_x^\infty u^{4m-2\nu+1}\,(u^2-x^2)^{-k+1}\,F_m\left(\nu-2m,\,\nu-2m+k-1,\,\frac{x^2}{u^2}\right)\psi(u)\,du$$

then

$$(3\cdot4)\quad \psi(x)=\frac{2\,\Gamma(\nu-m+k-1)\,\Gamma(\nu-m)}{\Gamma(m-k+2)\,\Gamma(\nu-2m+k-1)\,\Gamma(m+1)\,\Gamma(\nu-2m+k-2)}\times$$

$$\int_0^x {}_2F_1\left(\nu-m,\,-m\,;\,\nu-2m+k-2\,;\,\frac{u^2}{x^2}\right) d[u^{2\nu-4m+2k-4}\,\phi(u)]$$

$$+\frac{2\Gamma(\nu-m+k-1)\,\Gamma(\nu-m)\,x^{2\nu-2m}}{\Gamma(m-k+2)\,\Gamma(\nu-2m+k-1)\,\Gamma(-m+k-2)\,\Gamma(\nu+1)}\times$$

$$\int_x^\infty {}_2F_1\left(\nu-m,\,m-k+3\,;\,\nu+1\,;\,\frac{x^2}{u^2}\right) u^{2m-2\nu}\,d[u^{2\nu-4m+2k-4}\,\phi(u)]$$

provided

(*i*) m is a non-negative integer such that $\nu > m$, and $1 < k < 2$

(*ii*) $\int_0^\infty t^{2m-\nu+1}\,|\,\psi(t)\,|\,dt$ and

$$\int_0^\infty t^{2m-\nu-k+3}\left|\frac{d}{dt}\{t^{2\nu-4m+2k-4}\,\phi(t)\}\right|\,dt$$

are convergent,

(*iii*) $\frac{d}{dt}\{t^{2\nu-4m+2k-4}\,\phi(t)\}$ is continuous.

4. The proofs of these theorems are contained in the following lemmas.

Lemma I. Let

$$x^{2m-\nu}\,\phi(x) = A_m \int_0^\infty y\,J_\nu(xy)\,f(y)\,dy \tag{4·1}$$

and

$$\psi(x) = \int_0^\infty y\,J_{\nu-2m+k-2}(xy)\,y^{-k+2}f(y)\,dy \tag{4·2}$$

then

$$\phi(x) = \int_0^x u^{\nu-2m+k-1}(x^2-u^2)^{-k+1}\,F_m\left(\nu-2m,\ \nu-2m+k-1,\ \frac{u^2}{x^2}\right)\psi(u)\,du \tag{4·3}$$

provided

(*i*) m is a non-negative integer, such that $\nu > m-k+1$ and $k < 2$

(*ii*) $$\int_0^1 t^{\nu-2m+1}\,|f(t)|\,dt \text{ and } \int_1^\infty t^{-k+5/2}\,|f(t)|\,dt$$

are convergent,

(*iii*) $$\int_0^\infty t\,|\psi(t)|\,dt \text{ is convergent}$$

where $$A_m = \frac{\Gamma(m-k+2)\,\Gamma(\nu-2m+k-1)}{\Gamma(\nu-m+k-1)\quad 2^{k-1}}$$

Proof:

If the conditions (*i*) and (*ii*) are satisfied, then by applying Hankel's inversion theorem [4, p. 73] to (4·2), we have

$$y^{-k+2}f(y) = \int_0^\infty u\,J_{\nu-2m+k-2}(uy)\,\psi(u)\,du.$$

Substituting the value of $f(y)$ from the above relation into (4·1), we have

$$x^{2m-\nu}\,\phi(x) = A_m \int_0^\infty y^{k-1}J_\nu(xy)\,dy \int_0^\infty u\,J_{\nu-2m+k-2}(uy)\,\psi(u)\,du$$

$$= A_m \int_0^\infty u\,\psi(u)\,du \int_0^\infty y^{k-1}J_{\nu-2m+k-2}(uy)\,J_\nu(xy)\,dy$$

$$= A_m \int_0^\infty u^{1-k}\psi(u)\,du \int_0^\infty t^{k-1}J_{\nu-2m+k-2}(t)\,J_\nu\left(\frac{xt}{u}\right)dt$$

(4·3) is obtained by using (2·4). The change of the order of integration is justified under the conditions mentioned in the lemma.

Lemma II. Let $\phi(x)$ and $\psi(x)$ be defined as in (4·1) and (4·2),

then

$$\psi(x) = \frac{2\,\Gamma(\nu-m)\,\Gamma(\nu-m+k-1)\,x^{k-\nu-2}}{\Gamma(-m+k-1)\,\Gamma(\nu)\,\Gamma(\nu-2m+k-1)\,\Gamma(m-k+2)} \times \tag{4·4}$$

$$\int_0^x {}_2F_1\left(\nu - m, m - k + 2 ; \nu ; \frac{u^2}{x^2}\right) d[u^{2m}\ \phi(u)\ ] +$$

$$\frac{2\,\Gamma(\nu - m)\,\Gamma(\nu - m + k - 1)\,x^{\nu-2m+k-2}}{\Gamma(m)\,\Gamma(\nu - 2m + k - 1)\,\Gamma(m - k + 2)\,\Gamma(\nu - 2m + k - 1)} \times$$

$$\int_x^\infty u^{2m-2\nu}\ {}_2F_1\left(\nu - m, - m + 1 ; \nu - 2m + k - 1 ; \frac{x^2}{u^2}\right) d\,[u^{2m}\ \phi(u)\ ]$$

provided

(*i*) m is a non-negative integer, such that $\nu > m$, and $k > 1$.

(*ii*) $$\int_0^1 t^{\nu+1}\,|f(t)|\,dt \quad \text{and} \quad \int_1^\infty t^{3/2}\,|f(t)|\,dt$$

are convergent,

(*iii*) $$\int_0^\infty t^{1-\nu}\,\left|\frac{d}{dt}\,\{t^{2m}\ \phi(t)\}\right|\,dt \text{ is convergent,}$$

(*iv*) $$\frac{d}{dt}\,\{\,t^{2m}\ \phi(t)\,\} \text{ is continuous.}$$

Proof:

From (4·1), it can be easily derived that

$$\phi_1(x) = x^{-\nu}\,\frac{d}{dx}\{\,x^{2m}\ \phi(x)\,\}$$

$$= A_m \int_0^\infty y\,J_{\nu-1}(xy),\, y f(y)\,dy$$

If the conditions under (*ii*) are satisfied, then by applying Hankel's inversion heorem to the above relation, we have

$$y f(y) = \frac{1}{A_m}\int_0^\infty u\,J_{\nu-1}(uy)\ \phi_1(u)\,du$$

Substituting the value of $f(y)$ from the above relation into (4·2), we have

$$\psi(x) = \frac{1}{A_m}\int_0^\infty y^{-k+2}\,J_{\nu-2m+k-2}(xy)\,dy \int_0^\infty u\ \phi_1(u)\ J_{\nu-1}(uy)\,du$$

$$= \frac{x^{k-3}}{A_m}\int_0^\infty u\ \phi_1(u)\,du \int_0^\infty t^{-k+2}\,J_{\nu-2m+k-2}(t)\,J_{\nu-1}\left(\frac{ut}{x}\right)dt$$

(4·4) is obtained by using (2·5). The change of the order of integration is justified under the conditions mentioned in the lemma.

Lemma III. Let

(4·5) $$x^{\nu-2m+k-2}\ \phi_1(x) = A_m \int_0^\infty y\,J_{\nu-2m+k-2}(xy)\,f(y)\,dy$$

and

$$x^{2m-\nu}\,\psi(x) = \int_0^\infty y\,J_\nu(xy)\,y^{-k+2}f(y)\,dy \tag{4·6}$$

then

$$\phi(x) = \int_x^\infty u^{4m-2\nu+1}\,(u^2-x^2)^{-k+1}\,F_m\left(\nu-2m,\ \nu-2m+k-1,\ \frac{x^2}{u^2}\right)\psi(u)\,du \tag{4·7}$$

provided

(*i*) m is a non-negative integer, such that $\nu > m-k+1$ and $k < 2$,

(*ii*) $\int_0^1 t^{\nu-k+3}\,|f(t)|\,dt$ and $\int_1^\infty t^{-k+5/2}\,|f(t)|\,dt$

are convergent

(*iii*) $\int_0^\infty t^{2m-\nu+1}\,|\psi(t)|\,dt$ is convergent.

where

$$A_m = \frac{\Gamma(m-k+2)\,\Gamma(\nu-2m+k-1)}{\Gamma(\nu-m+k-1)\quad 2^{k-1}}$$

Proof:

If the conditions (*i*) and (*ii*) are satisfied, then by applying Hankel's inversion theorem to (4·6), we have

$$y^{-k+2}f(y) = \int_0^\infty u\,J_\nu(uy)\quad u^{2m-\nu}\,\psi(u)\,du$$

Substituting the value of $f(y)$ from the above relation into (4·5), we have

$$u^{\nu-2m+k-2}\ \phi(x) = A_m\int_0^\infty y^{k-1}J_{\nu-2m+k-2}(xy)\,dy\ \int_0^\infty u\,J_\nu(uy)\,u^{2m-\nu}\,\psi(u)\,du$$

$$= A_m\int_0^\infty u^{1+2m-\nu}\,\psi(u)\,du\int_0^\infty y^{k-1}J_{\nu-2m+k-2}(xy)\,J_\nu(uy)\,dy$$

$$= \frac{A_m}{x^k}\int_0^\infty u^{1+2m-\nu}\,\psi(u)\,du\int_0^\infty t^{k-1}J_{\nu-2m+k-2}(t)\,J_\nu\left(\frac{ut}{x}\right)dt$$

(4·7) is obtained by using (2·4). The change of the order of integration is justified under the conditions mentioned in the lemma.

Lemma IV. Let $\phi(x)$ and $\psi(x)$ be defined as in (4·5) and (4·6), then

$$\psi(x) = \frac{2\Gamma(\nu-m+k-1)\,\Gamma(\nu-m)}{\Gamma(m-k+2)\,\Gamma(\nu-2m+k-1)\,\Gamma(m+1)\,\Gamma(\nu-2m+k-2)}\times \tag{4·8}$$

$$\int_0^x {}_2F_1\left(\nu-m,\,-m\,;\,\nu-2m+k-2\,;\,\frac{u^2}{x^2}\right) d[u^{2\nu-4m+2k-4}\,\phi(u)]$$

$$+\frac{2\Gamma(\nu-m+k-1)\,\Gamma(\nu-m)\,x^{2\nu-2m}}{\Gamma(m-k+2)\,\Gamma(\nu-2m+k-1)\,\Gamma(-m+k-2)\,\Gamma(\nu+1)}\times$$

$$\int_x^\infty u^{2m-2\nu}\,{}_2F_1\left(\nu-m,\,m-k+3\,;\,\nu+1\,;\,\frac{x^2}{u^2}\right) d[u^{2\nu-4m+2k-4}\,\phi(u)]$$

provided

(*i*) m is a non-negative integer, such that $\nu > m$ and $k > 1$

(*ii*) $\int_0^1 t^{\nu-2m+k-1}\,|f(t)|\,dt$ and $\int_1^\infty t^{3/2}\,|f(t)|\,dt$ are convergent

(*iii*) $\int_0^\infty t^{2m-\nu-k+3}\left|\frac{d}{dt}\{t^{2\nu-4m+2k-4}\,\phi(t)\}\right|dt$ is convergent,

(*iv*) $\frac{d}{dt}\{t^{2\nu-4m+2k-4}\,\phi(t)\}$ is continuous

Proof:

From (4·5), it can be easily derived that

$$\phi_2(x) = x^{-\nu+2m-k+2}\,\frac{d}{dx}\{x^{2\nu-4m+2k-4}\,\phi(x)\}$$

$$= A_m\int_0^\infty y\,J_{\nu-2m+k-3}(xy)\,.\,y f(y)\,dy$$

If the conditions under (*ii*) are satisfied, then by applying Hankel's inversion theorem to the above relation, we have

$$y\,f(y) = \frac{1}{A_m}\int_0^\infty u\,J_{\nu-2m+k-3}(uy)\,\phi_2(u)\,du$$

Substituting the value of $f(y)$ from the above relation into (4·6), we have

$$x^{2m-\nu}\,\psi(x) = \frac{1}{A_m}\int_0^\infty y^{-k+2}\,J_\nu(xy)\,dy\int_0^\infty u\,\phi_2(u)\,J_{\nu-2m+k-3}(uy)\,du$$

$$= \frac{1}{A_m}\int_0^\infty u^{k-2}\,\phi_2(u)du\int_0^\infty t^{-k+2}\,J_{\nu-2m+k-3}(t)\,J_\nu\left(\frac{xt}{u}\right)dt$$

(4·8) is obtained by using (2·6). The change of the order of integration is justified under the conditions mentioned in the lemma.

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References

1. Srivastava K. N. On some integral transforms involving Jacobi functions. *Annales, Polonici Mathematici*, **16**, 1965.
2. Magnus, W. and Oberhettinger, F. *Special functions of Mathematical Physics, New York*, 1949.
3. Rainville, E. D. Special functions. *The Macmillan Company, New York*, 1963.
4. Erdelyi, A. Higher transcendental functions. Vol. **2**, *McGraw-Hill*, 1953.
5. Tranter, C. J. Integral transforms in mathematical physics. *John Wiley and Sons, Inc., New York*, 1962.
6. ——— *Quart. J. Mech. App. Math.*, **3**, 411, 1950.
7. ——— *Quart. J. Mech. App. Math.*, 317, 1954.
8. ——— *Quart. J. Math. Oxford*, **10**, 110, 1959.
9. Srivastava, K. N. Dual integral equations with Bessel function and trigonometrical kernels. *Proc. Edin. Math. Soc., Vol.* 13, (*series II*), *Part, 4, Dec.* 1963.

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On symmetrical thermal bending of piezoelectric plates

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Abstract

The present note is an attempt to solve the problem of thermal bending of a piezoelectric plate made of quartz. The equations of mechanical equilibrium, equations of electricity together with the constitutive relations of the piezoelectric material have been made use of in investigating the symmetrical bending of the plate The problem envisaged here has been reduced to an isothermal problem and in particular, the problem has been solved for a certain thermal distribution.

Introduction

In recent years, the piezoelectric problems so far attempted are, in most cases, analogues of corresponding purely anisotropic plates and in fact, a large number of piezoelectric problems of statical interest are of this type. In this context, the papers of Paria[1], Paul[2], Sinha[3,4,5], Giri[9], Das[11], Roy[13] may be cited. The present note is, in essence, an attempt of this kind and it seeks to investigate the interaction of three fields *viz.* mechanical, electrical and thermal in a piezoelectric quartz. The investigation centres on the nature of symmetrical bending in piezoelectric plates and is carried out on the lines of the papers on purely anisotropic problems by Herrman[14], Baclig and Conway[15], Carrier[16], and of the treatise by Boley and Weiner[17]. It is pertinent to refer, in this connection, to the article of Mindlin[18] who is, perhaps, the first to formulate the equations of vibration for piezoelectric plates for certain modes of vibration. The constitutive relations for a piezolectric crystal in which thermal fields are accomodated have been derived by Mindlin[20]. As far as the present author is aware, there has been no attempt to solve the problem set forth here, though the cross-fertilization of three fields has been considered in some papers by Sinha[4,6,7,8], Giri[10], Das[12]. In what follows, it is shown that the problem can be reduced to the problem of isothermal elasticity. The method presented here has been illustrated by supposing a radial distribution of temperature.

Fundamental Equations

The constitutive relations in polar coordinates (r, θ) for a piezoelectric quartz, as in Mason[18] are given by

$$
\begin{aligned}
S_r &= s_{11}\,\sigma_r + s_{12}\,\sigma_\theta + d_{11}\,E_r + \mu_1\,T \\
S_\theta &= s_{12}\,\sigma_r + s_{11}\,\sigma_\theta + d_{11}\,E_r + \mu_2\,T \\
S_{r\theta} &= s_{14}\,\tau_{r\theta} \\
D_r &= d_{11}\,(\sigma_r - \sigma_\theta) + \varepsilon_{11}\,E_r + p_1\,T \\
D_\theta &= -\,d_{14}\,\tau_{r\theta} + p_1\,T
\end{aligned} \qquad (1)
$$

where $S's$ are the strain components, $\sigma's$ are the stress components, $\tau_{r\theta}$ is the shearing stress, E_r is the electric intensity component, T is the temperature, $s's$ are the elastic compliances, d_{11} the piezoelectric constant, ϵ_{11} the dielectric permittivity, $\mu's$ and $p's$ the thermopiezoelectric moduli. It is evident from the foregoing equations (1), that the electric field is characterized by its radial component. These equations, expressed in terms of stresses, are

$$(2)\qquad \sigma_r = \frac{1}{(s_{11}^2 - s_{12}^2)}\left[(s_{11} S_r - s_{12} S_\theta) - (s_{11} - s_{12})\, d_{11} E_r + (s_{12}\mu_2 - s_{11}\mu_1)\, T \right]$$

$$(3)\qquad \sigma_\theta = \frac{1}{(s_{11}^2 - s_{12}^2)}\left[(s_{11} S_\theta - s_{12} S_r) - (s_{11} - s_{12})\, d_{11} E_r + (s_{12}\mu_1 - s_{11}\mu_3)\, T \right]$$

$$(4)\qquad \tau_{r\theta} = \frac{1}{s_{14}} S_{r\theta}$$

$$(5)\qquad D_r = \frac{d_{11}}{(s_{11}^2 - s_{12}^2)} (s_{11} + s_{12})(S_r + S_\theta) + \left[p_1 - \frac{(s_{11} + s_{12})(\mu_1 - \mu_2)\, d_{11}}{(s_{11}^2 - s_{12}^2)} \right] T + \epsilon_{11} E_r$$

$$(6)\qquad D_\theta = - \frac{d_{14}}{s_{14}} S_{r\theta} + p_1 T.$$

As in isotropic plate theory, we assume that the effects of stress components on the deflection are small and that sections parallel to the middle surface remain plane. If l is the length of the thickness of the plate, as in Mindlin[19], we introduce the couples per unit length given by

$$(7)\qquad M_r = \int_{-l/2}^{+l/2} (\sigma_r)\, z dz,\ M_\theta = \int_{-l/2}^{+l/2} \sigma_\theta .\, z dz,\ M_{r\theta} = - M_{\theta r} = \int_{-l/2}^{+l/2} \tau_{r\theta} .\, z dz$$

Let us also introduce M_T defined by

$$(8)\qquad M_T = \int_{-l/2}^{+l/2} T .\, z dz.$$

Since the neutral surface $z = o$ remains displaced horizontally, and the bending is symmetrical, the values of u, v at any point are given by

$$(9)\qquad u = - z \frac{\partial w}{\partial r},\ v = o$$

where w is the deflection of the middle surface of the disk. The strains are defined by

$$(10)\qquad \begin{cases} S_r = - z \dfrac{\partial^2 w}{\partial r^2} \\ S_\theta = - \dfrac{z}{r} \dfrac{\partial w}{\partial r} \\ S_{r\theta} = o \end{cases}$$

Also, $Er = \frac{\partial V}{\partial r}$ (V, the electrostatic potential).

From (7) and (10), we have

$$M_r = -\frac{s_{11}}{(s_{11}^2 - s_{12}^2)}\frac{\partial^2 \omega}{\partial r^2} + \frac{s_{12}}{s_{11}^2 - s_{12}^2}\frac{1}{r}\frac{\partial \omega}{\partial r} - \frac{(s_{11} - s_{12})\, d_{11} l}{(s_{11}^2 - s_{12}^2)}\frac{\partial V}{\partial r} + \frac{(s_{12}\mu_2 - s_{11}\mu_1)}{(s_{11}^2 - s_{12}^2)} M_T \tag{11}$$

$$M_\theta = -\frac{s_{11}}{(s_{11}^2 - s_{12}^2)}\frac{1}{r}\frac{\partial w}{\partial r} + \frac{s_{12}}{(s_{11}^2 - s_{12}^2)}\frac{\partial^2 w}{\partial r^2} - \frac{(s_{11} - s_{12})\, d_{11} l}{(s_{11}^2 - s_{12}^2)}\frac{\partial V}{\partial r} + \frac{(s_{12}\mu_2 - s_{11}\mu_1)}{(s_{11}^2 - s_{12}^2)} M_T - \left\{\frac{(s_{11} + s_{12})(\mu_1 - \mu_2)}{(s_{11}^2 - s_{12}^2)} - p_1\right\}\frac{\partial M_T}{\partial r} \tag{12}$$

$$M_{r\theta} = 0 \tag{13}$$

The equations of equilibrium have the form given by

$$\frac{\partial M_r}{\partial r} + \frac{M_r - M_\theta}{r} - Q_r = 0 \tag{14}$$

$$\frac{\partial}{\partial r}(r\, Q_r) = 0 \tag{15}$$

where we take Q_r to be the only non-vanishing component of the shearing force per unit length.

Apart from this, we have the Gauss's divergence equation which gives

$$\frac{\partial D_r}{\partial r} = 0 \tag{16}$$

The equations (2) – (16) constitute the fundamental equations.

Governing Differential Equation and its solution

From equations (5), (11), (12), (16), we have on, simplification,

$$\frac{d_{11}(s_{11} + s_{12})}{(s_{11}^2 - s_{12}^2)}\left[\frac{1}{r}\frac{\partial w}{dr} - \frac{\partial^2 w}{\partial r^2}\right] - \frac{(s_{11} + s_{12})(\mu_1 - \mu_2)}{(s_{11}^2 - s_{12}^2)} M_T + \varepsilon_{11} l \frac{\partial V}{\partial r} + p_1 T = k \text{ (constant) (say.)} \tag{17}$$

Using (11) and (12), we get from (14),

$$Q_r = -\frac{s_{11}}{(s_{11}^2 - s_{12}^2)}\frac{\partial^3 w}{\partial r^3} - \frac{s_{11}}{(s_{11}^2 - s_{12}^2)}\frac{1}{r}\frac{\partial^2 w}{\partial r^2} + \frac{s_{11}}{(s_{11}^2 - s_{12}^2)}\cdot\frac{1}{r}\frac{\partial w}{\partial r} \frac{(s_{11} - s_{12})\, d_{11} l}{(s_{11}^2 - s_{12}^2)}\frac{\partial^2 V}{\partial r^2} + \frac{(s_{12}\mu_2 - s_{11}\mu_1)}{(s_{11}^2 - s_{12}^2)}\frac{\partial M_T}{\partial r} + \frac{(s_{11} + s_{12})(\mu_2 - \mu_1)}{(s_{11}^2 - s_{12}^2)}\frac{M_T}{r} \tag{18}$$

The equation (15), on making use of (17) and (18), yields

$$\left(c_1\frac{\partial^4 w}{\partial r^4} + \frac{2c_1}{r}\frac{\partial^3 w}{\partial r^3} - \frac{c_2}{r^2}\frac{\partial^2 w}{\partial r^2} + \frac{c_3}{r^3}\frac{\partial w}{\partial r}\right) = -\left(c_4\frac{\partial^2 M_T}{\partial r^2} + \frac{c_5}{r}\frac{\partial M_T}{\partial r}\right) \tag{19}$$

where c_1, c_2, c_3, c_4, c_5 are all constants containing the material parameters of the problem. This equation, in the absence of electric field, essentially agrees with

the equation for anisotropic plates. Let us now assume the distribution of the temperature at any point to be given by

$$T = T_0 z^{r^n}$$

where T_0 and n are constants.

$$\therefore M_T = \frac{T_0 l^3}{12} r^n$$

Hence (19) becomes

$$(20) \qquad \left(c_1 \frac{\partial^4 w}{\partial r^4} + \frac{2c_1}{r} \frac{\partial^3 w}{\partial r^3} - \frac{c_2}{r^2} \frac{\partial^2 w}{\partial r^2} + \frac{c_3}{r^3} \frac{\partial w}{\partial r} \right) = - n\{c_4(n-1) + c_5\} \frac{T_0 l^3}{12} r^{n-2}$$

Solving this equation we have

$$(21) \qquad w = A (\log r)^{m_1} + B (\log r)^{m_2} + C (\log r)^{m_3} + D (\log r)^{m_4} - \frac{n\{c_4(n-1) + c_5\} T_0 l^3 r^{n+2}}{c_1(n+2)^4 - 4c_1(n+2)^3 + (5c_1 - c_2)(n+2)^2 + (c_2 + c_3)(n+2)}$$

where A, B, C, D are constants of integration. To determine them, let us consider an annular plate whose peripheries $r = r_1$ and $r = r_2$ are clamped so that $w = \frac{dw}{dr} = 0$ both at $r = r_1$ and $r = r_2$. It is a formal algebra to find out the four quantities A B, C, D from these conditions and thus we can evaluate the deflection and from it the components of the bending moment in terms of the parameters of the problem. It may be mentioned that the constant in (17) may be determined by maintaining a reasonable electric voltage between the inner and outer boundaries. It is to be pointed out that while in a purely anisotropic plate, the deflection is affected partly to the extent of a constant, it is not in the case of a piezoelectric plate in which case it is characterized by a logarithmic variation of the radial distance.

References

1. Paria, G. *Jour. Sc. Engg. Res.*, **4**(2) : 381, 1960.
2. Paul, H. S. *Austr. Jour. Appl. Sc.*, **1** : 98, 1962.
3. Sinha, D. K. *Rev. Mec. Appliquee*, **VII** : 947, 1962.
4. Sinha, D. K. *Jour. Acoust. Soc. Amer.* **36** : 1926, 1964.
5. Sinha, D. K. *Proc. Natl. Inst. Sc. India*, **30-A** : 650, 1964.
6. Sinha, D. K. *Ind. Jour. Mech. Math.*, **11** : 10, 1964.
7. Sinha, D. K. *Proc. Ind. Acad. Scs.*, **XLV** : 136, 1967.
8. Sinha, D. K. *Ind. Jour. Maths.*, **8** : 15, 1966.
9. Giri, R. R. *Ind. Jour. Pure and Appld. Phys.*, **4** : 224, 1966.
10. Giri, R. R. *Proc. Natl. Inst. Sc. India*, **35-A** : 325, 1967.
11. Das, N. C. *Jour. Sc. Engg. Res.* **X**(I) : 167, 1966.
12. Das, N. C. *Proc. Natl. Inst. Sc. India*, **35-A** : 173, 1969.
13. Roy, P. *Jour. Sc. Engg. Res.*, **IX**(2) : 324, 1965.
14. Herrmann, J. *Jour. Frank. Inst.*, 421, 1955.
15. Baclig, E. S. and Conway, H. D. *Jr. Appld. Mechs.*, **23** : 11, 1956.
16. Carrier, G. F. *Applied Anisotropic Elasticity.*
17. Boley, P. and Weiner, N. Theory of Thermal Stress, *John Wiley & Sons.*
18. Mason, W. P. Piezoelectric Crystals and their Applications to Ultrasonics. *D. Van Nostrand Co., New York*, 1948.
19. Mindlin, R. D. *Jr. Appl. Phys.*, **23** : 83, 1952.
20. Mindlin, R. D. Problems of Continuum Mechanics, *Pennsylvania*, 282, 1961.

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Chemical Examination of the Fat from the Root of *Moringa concanensis*

By

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Abstract

On chemical examination, the fat from the root of *Moringa concanensis* showed the presence of stearic acid 33·67% palmitic acid 38·5% and oleic acid 26·625%. The unsaponifiable matter has been found to contain ceryl alcohol and β-sitosterol.

The plant *Moringa concanensis*[1,2] (Hindi : Sajana) belongs to N. O. *Moringaceae*. The root is tonic to body and lungs. It is reported[2] to be useful in bronchitis, piles, rheumatism and loss of appetite and it also cures inflamation of heart and chest.

Experimental

5 Kg of the dried and powdered roots were extracted with petroleum ether (40-60°) in a soxhlet extractor. On keeping overnight in the refrigerator the extract gave no deposit. The solvent was then distilled off under reduced pressure and the fat (yield 80 g.) thus obtained gave the following constants—

Acid value = 4·2	Saponification value = 201·6
Iodine value = 25·3	Unsaponifiable matter = 1·8%

65 g. of the fat was saponified with 0·5 N etanolic KOH solution. The fatty acids and unsaponifiable matter were recovered by the usual procedure.

The mixed fatty acids thus obtained were then segregated into solid and liquid fractions by Twitchell Lead-salt Alcohol process[3], modified by Hilditch and coworkers[4]. They were found to have the following constants—

Fatty acid	Amount	Percentage	Iodine value	Saponification value
Liquid	22 g.	33·84%	50·6	201·6
Solid	41 g.	60·07%	1·2	189·3

Segregation of solid fatty acids (By urea adduct method)—Ten 250 ml conical flasks with glass stoppers numbered S_1 to S_{10} were taken. The flasks S_2 to S_9 were employed for adduct formation, while the flask S_{10} for collecting raffinates. Adduct formation was carried out by the procedure followed by Tandon and Tiwari[8]. The fatty acids were recovered from each flask by treating the adduct in the respective flask with warm acidulated water followed by extraction with ether. The different fractions were weighed separately and their saponification and iodine values were determined. The results are recorded in Table 1.

A qualitative separation of solid fatty acids was done by paper chromatography. A Whatman No. 1 filter paper (24 × 30 cm) was impregnated with a 10% solution of liquid paraffin in benzene, the paper was spotted with unknown sample along with the authentic samples of palmitic acid, stearic acid, behenic acid, arachidic acid and oleic acid. Glacial acetic acid was allowed to run over the paper for 48 hours. It was then dried in an electric oven and dipped into a 5% solution of copper acetate. The chromatogram was washed with distilled water and then dipped into a 6% solution of potassium ferrocyanide[6,7,8]. The unknown spots corresponded with the spots of palmitic acid, stearic acid and oleic acid.

TABLE 1

Flask No.	Saponification value	Iodine value	Saponification equivalent	Amount g.	Palmitic g.	Stearic g.	Oleic g.
S_1	214·2	1·2	261·6	3·19	1·95	1·20	·04
S_2	209·8	0·8	267·1	1·5	1·145	0·23	·013
S_3	211·0	0·6	265·1	2·0	0·665	1·322	·013
S_4	200·1	1·1	278·8	1·6	0·420	1·162	·018
S_5	204	0·8	274·5	1·3	0·44	0·84	·012
S_6	205·8	1·1	271·8	0·5	0·199	0·295	·006
S_7	208·0	0·7	270·0	0·7	0·324	0·37	·006
S_8	206·0	0·58	271·8	1·2	0·531	0·662	·007
S_9	208·0	0·62	270·0	0·6	0·280	0·316	·004
S_{10} (Raffinates)	210	3·2	259·2	2·1	1·8665	0·1595	·074
Total weight 14·699 g.					7·820	6·558	·193
Percentage of acids in solid fraction					52·54%	44·66%	1·32%

Segregation of liquid fatty acids.—Eighteen conical flasks numbered $L_1 - L_{18}$, each of 250 ml capacity and having ground glass stoppers, were taken. The flasks L_1 to L_9 were employed for adduct formation and the flasks L_{10} to L_{18} for collecting raffinates. The process of fractionation was the same as described in the case of solid fatty acid fraction except that in this case the raffinates were collected separately. The acids in different fractions were recovered in the same way as described with solid fatty acid fractions. Their saponification and iodine values were determined. The results are recorded in Table 2.

A qualitative separation of individual fatty acids was done by paper chromatography using Whatman No. 1 filter paper (12 × 30 cm). The paper was impregnated with a 10% solution of liquid paraffin in benzene. After drying, the paper was spotted with unknown sample along with the authentic samples of oleic, linoleic, palmitic and stearic acids. 75% ethanol solution was allowed to run over the paper for 24 hours. The chromatogram was dried and kept in an iodine chamber[9]. Only one brown spot was observed. Excess of iodine was removed by washing the paper with distilled water and then the paper was dipped into a 1% solution of silver nitrate and finally into a dilute solution of sodium sulphide. Only one black spot corresponding to oleic acid, was obtained.

TABLE 2

Flask No.	Saponification value	Iodine value	Saponification equivalent	Amount	Palmitic acid	Stearic acid	Oleic acid
L_1	202·2	38·3	277	3·80g	0·65g	1·55g	1·60g
L_2	200·0	40·2	280	2·6g	0·23g	1·22g	1·15g
L_3	210·0	45·0	266	2·6g	1·09g	0·21g	1·30g
L_4	212·0	49·3	263	2·0g	0·707g	0·213g	1·08g
L_5	211·0	61·0	265	1·6g	0·43g	3·09g	1·08g
L_6	199·3	46·2	281	1·3g	0·06g	0·576g	·664g
L_7	205·1	57·8	273·1	0·8g	0·112g	0·182g	·506g
L_8	208·2	60·0	269·2	0·6g	0·105g	0·102g	·403g
L_9	208·0	39·1	269·1	0·8g	0·340g	0·093g	·367g
Raffinates.							
L_{18}	204·0	49·0	273·6	0·80g	0·351g	·014g	0·435g
L_{17}	215·0	63·0	259	0·88g	0·230	·030	0·620g
L_{16}	211·0	62·0	265	0·62g	0·167	·088	0·365g
L_{15}	209·8	41·0	267	0·56g	0·276g	·034	0·250
L_{14}	–	60·0	–	0·25	–	–	0·16
L_{13}	–	48·0	–	0·18	–	–	0·96
Total weight 19·40g					4·748	4·402	10·076
% acid in liquid fraction					24·46%	22·68%	51·93%

Study of the unsaponifiable matter

The unsaponifiable matter was submitted to chromatographic separation over silica gel column. The column was eluted with petroleum ether and benzene.

The fractions obtained by eluting with petroleum ether were spotted on a T. L. C. silica gel G. plate and the plate was developed in petroleum ether : benzene (9 : 1) mixture. The developed plate was sprayed with 5% H_2SO_4 in ethanol and heated to 200° in an electric oven for 10 minutes. Only one spot of each fraction appeared. Rf. values of all the spots were the same showing thereby that all the fraction contained the same compound. The above fraction were then mixed together and the solvent evaporated. The crude compound was then crystallised from methanol, when fine needles (m.p. 134°) were obtained. This compound underwent Liebermann-Burchard reaction indicating it to be a sterol.

Found	$C_{29}H_{50}O$ requires
C = 81·50%	C = 81·05%
H = 11·32%	H = 12·07%
Molecular weight = 404	Molecular weight = 414
(Semi-micro-Rast)	

On acetylation with acetic anhydride and pyridine a monoacetate melting at 127--28° was obtained.

Found	Calculated for $C_{31}H_{52}O_2$
C = 81·62%	C = 81·57%
H = 11·8%	H = 11·4%

The peaks in the infra red spectrum were found at 3500 cm^{-1} 1640 cm^{-1}, 1470 cm^{-1}, 1445 cm^{-1} which showed its identity with β-sitosterol.

The fractions obtained by eluting with benzene were spotted on a T. L. C. silica gel plate and the plate was developed in benzene : methanol (25 : 75) mixture. The developed plate was sprayed with 5% H_2SO_4 in ethanol and heated to 200° in an electric oven for 10 minutes. Spots having the same Rf values for each fraction were obtained, showing that all the above fractions were the same. They were then mixed together. The solvent was evaporated when a residue was obtained.

The residue on repeated crystallisation from benzene yielded a pure compound m.p. 78°. This compound neither contained nitrogen, halogen, sulphur or phosphorous nor it gave the reactions of acid, aldehyde, ketone, lactone or sterol, but gave a light red coloration with a solution of cerric ammonium nitrate, showing the presence of an alcoholic group in it.

Found	$C_{27}H_{56}O$ requires
C = 81·52%	C = 81·67%
H = 13·48%	H = 14·13%
Molecular weight 308 (Semi-micro Rast)	Molecular weight = 312

If furnished a monoacetate m.p. 67°. It was identified as ceryl alcohol by mixed melting point and infra red spectrum. Peaks in the infra red spectrum were obtained at 3498 cm^{-1}, 2885 cm^{-1}, 2800 cm^{-1}, 1470 cm^{-1} and 730 cm^{-1}.

References

1. Chopra, R. N., Nayar, S. L. and Chopra, I. C. Glossary of Indian Medicinal plants. *C. S. I. R. Pub.* 1965 ed. p. 170.
2. Kirtikar, K. R. and Basu, B. D. Indian Medical Plants Vol. I. p. 682.
3. Twitchell, E. J. *Ind. eng. chem.*, **13**, 806, 1921.
4. Hilditch. The chemical constituent of natural fats. Chapman and Hall 3rd Ed.
5. Inovye, X. and Noda, M. *J. Ag. Chem. Soc. Japn.*, **26**, 634.
6. *Ibid.*, **27**, 50, 1953.
7. Spiteri. *J. Bull. Soc. Chem. biol.*, **36**, 1355, 1954.
8. Tandon, S. P. and Tiwari, K. P. Chemical examination of the plant Fagonia cretica Linn : Study of Fat. *Proc. Nat. Acad. Sci. India*, **A36**(3), (In press).
9. Srivastava, P. K., Bhute, P. G and Mrs. Lela, S. A., *I. O. and S. J.* **29**, 243, 1964.

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Some Time Reversal Problems of Heat Conduction

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Abstract

Time reversal heat conduction problems in a thermodynamical system with known boundary conditions have been solved with the help of integral transforms for cylindrical shell of infinite height with heat generation and radiation, truncated wedge of finite height and semi-infinite solid containing an exterior plane crack with a circular boundary and infinitely long cylindrical cavity.

1. Introduction

Given the temperature distribution at time $T(T > 0)$ and the suitable boundary conditions, the determination of initial temperature distribution throughout a medium is known as 'time reversal problem'. This type of problem has a special significance in finding the temperature distribution at any time and position, when the temperature at a particular instant $T\,(T > 0)$ is known.

The solution of the time reversal heat conduction problem play an important role in the thermodynamic system with known boundary conditions. When such a system has reached maximum possible entropy, it is not possible to trace back and find the initial conditions, because the change in the entropy of a substance depends only on its initial and final conditions and on the particular reversible process by which it passes from one state to another [3, p. 311]. Temperature distribution at the intermediate state between the initial and steady state can lead us to the initial conditions [2].

Sabherwal [4] has tackled some time reversal problems without any heat source and radiation with the help of "Operational methods".

In this communication, I have solved three time reversal problems, with respect to thermodynamic system, of transient heat conduction by considering the flow of heat in

(*i*) Cylindrical shell of infinite height with heat generation and radiation,

(*ii*) truncated wedge of finite height, and

(*iii*) semi-infinite solid containing an exterior plane crack with a circular boundary and an infinitely long cylindrical cavity.

2. Required Integral Transforms

Marchi and Zgrablich [1] have defined the finite integral transform of a function $f(x)$ in the form

$$\bar{f}(n) = \int_a^b x\, f(x)\, s_p\,(k_1, k_2, \mu_n\, x)\, dx \tag{2·1}$$

where μ_n is chosen as a positive root of the equation

(2·2) $$J_p(k_1, \mu a)\, G_p(k_2, \mu b) - J_p(k_2, \mu b)\, G_p'(k_1, \mu a) = 0$$

and the inversion transform is given by

(2·3) $$J(x) = \sum_n \frac{1}{c_n} \bar{f}(n)\, s_p(k_1, k_2, \mu_n x)$$

the summation being over the positive roots of (2·2) and c_n is given by

(2·4) $$c_n = \tfrac{1}{2} b^2 \{ \bar{s}_p^2 (k_1, k_2, \mu_n b) - \bar{s}_{p-1} (k_1, k_2, \mu_n b)\ \bar{s}_{p+1} (k_1, k_2, \mu_n b) \} \times$$
$$- \tfrac{1}{2} a^2 \{ \bar{s}_p^2 (k_1, k_2, \mu_n a) - \bar{s}_{p-1} (k_1, k_2, \mu_n a)\ \bar{s}_{p+1} (k_1, k_2, \mu_n a) \}$$

with

(2·5) $$\bar{s}_q (k_1, k_2, \mu_n x) = J_q (\mu_n x) [G_p (k_1, \mu_n a) + G_p (k_2, \mu_n b)] - G_q (\mu_n x) \times [J_p (k_1, \mu_n a) + J_p (k_2, \mu_n b)]$$

and

(2·6) $$s_p (k_1, k_2, \mu_n x) = J_p (\mu_n x) [G_p (k_1, \mu_n a) + G_p (k_2, \mu_n b)] - G_p(\mu_n x) \times [J_p (k_1, \mu_n a) + J_p (k_2, \mu_n b)]$$

where

$$J_p (k_i, \mu x) = J_p (\mu x) + k_i \mu J_p' (\mu x)$$
$$G_p (k_i, \mu x) = G_p (\mu x) + k_i \mu G_p' (\mu x)\ (i = 1, 2)$$

and $J_p (\mu x)$ and $G_p (\mu x) = \frac{1}{2} \pi \operatorname{cosec} (p \pi) [J_{-p} (\mu x) - e^{-ip\pi} J_p (\mu x)]$ are Bessel functions of first and second kind, respectively, of order p.

Finite sine transform of a function $f(x)$ is defined as

(2·7) $$\bar{F} (m) = \int_0^h f(x) \sin \frac{m\pi x}{h}\, dx$$

where $f(x)$ is given by [5, p. 74]

(2·8) $$f(x) = \frac{2}{h} \sum_{m=1}^{\infty} \bar{F} (m) \sin \frac{m\pi x}{h}$$

We shall also use the integral transform of a function $f(x)$ defined as [7, p. 89]

(2·9) $$f'(\eta) = \int_a^b x f(x)\, B_n (\eta x)\, dx,\ b > a,$$

where $$B_n (\eta x) = J_n (\eta x)\, Y_n(\eta a) - Y_n (\eta x)\, J_n(\eta a),$$

and $Y_n (\eta x)$ is the Bessel function of the second kind of order n, and η is chosen as a positive root of the equation

(2·10) $$B_n (\eta b) = 0$$

The inversion formula is [7, p. 89]

$$(2\cdot11) \qquad f(x) = \sum_{\eta} \frac{\pi^2}{2} \frac{\eta^2 J_n^2(\eta b) B_n(\eta x)}{[J_n^2(\eta a) - J_n^2(\eta b)]} f'(\eta)$$

the summation being over the positive roots of (2·10).

The Weber transform, $f_w(s)$, of a continuous real function $f(x)$ is defined on $(1, \infty)$ by

$$(2\cdot12) \qquad f_w(s) = \int_1^{\infty} x f(x) [J_0(xs) Y_0(s) - Y_0(xs) J_0(s)] dx$$

where J_0 and Y_0 are Bessel functions of the first and second kind. It easily follows [6, p. 75] that

$$(2\cdot13) \qquad f(x) = \int_0^{\infty} s \frac{J_0(xs) Y_0(s) - Y_0(xs) J_0(s)}{[J_0^2(s) + Y_0^2(s)]} f_w(s) ds$$

And Fourier sine transform of a function $f(x)$ is defined as [7, p. 15]

$$(2\cdot14) \qquad f'(\xi) = \int_0^{\infty} f(x) \sin(\xi x) dx$$

then [7, p. 15]

$$(2\cdot15) \qquad f(x) = \frac{2}{\pi} \int_0^{\infty} f'(\xi) \sin(\xi x) d\xi.$$

3. Heat flow in a cylindrical shell of infinite height with heat generation and radiation

We consider the diffusion of heat in a cylindrical shell $(a \leqslant r \leqslant b)$ when there are sources of heat within it which lead to an axially symmetrical temperature distribution. If we assume that the rate of generation of heat is independent of the temperature and that the shell is infinitely long, then the fundamental differential equation shall be of the form

$$(3\cdot1) \qquad \frac{\partial u}{\partial t} = \alpha \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right) + \underset{\sim}{(H)}(r, t), \; a \leqslant r \leqslant b, \; 0 < t,$$

where α is a constant known as diffusivity, with the physical conditions

$$(3\cdot2) \qquad u(a, t) + k_1 \frac{\partial u}{\partial r}(a, t) = F_a(t), \; 0 < t,$$

$$(3\cdot3) \qquad u(b, t) + k_2 \frac{\partial u}{\partial r}(b, t) = F_b(t), \; 0 < t,$$

where k_1 and k_2 are the radiation constants on the two surfaces,

$$(3\cdot4) \qquad u(r, T) = f(r) \text{ (known), for all } a < r < b,$$

$$(3\cdot5) \qquad u(r, o) = s(r) \text{ (unknown), for all } a < r < b,$$

where $F_a(t)$ and $F_b(t)$ are known functions of time.

Let

$$\bar{u}(n, t) = \int_a^b r\, u(r, t)\, s_0(k_1, k_2, \mu_n r)\, dr$$

where μ_n are the positive roots of the equation (2·2), when $p = 0$.

Now by the application of the transformation defined in (2·1) to (3·1), with respect to r, when $p = 0$, and when we use the conditions (3·2) and (3·3) we obtain

$$\frac{d\bar{u}}{dt} + \alpha\, \mu_n^2\, \bar{u} = \psi(t) \tag{3·6}$$

where

$$\psi(t) = \overline{(\widetilde{H})}\ (n, t) + \frac{ab}{k_2}\, s_0(k_1, k_2, \mu_n b)\, F_b(t) - \frac{\alpha a}{k_1}\, s_0(k_1, k_2, \mu_n a)\, F_a(t) \tag{3·7}$$

and

$$\overline{(\widetilde{H})}(n, t) = \int_a^b r\, (\widetilde{H})\ (r, t)\, s_0(k_1, k_2, \mu_n r)\, dr \tag{3·8}$$

The solusion of the differential equation (3·6) is

$$\bar{u}(n, t) = e^{-\alpha\, \mu_n^2 t} \int_0^t \psi(t')\ e^{\alpha\, \mu_n^2 t'}\, dt' + A\, e^{-\alpha \mu_2 {}_n t}, \tag{3·9}$$

where A is an arbitrary constant. Using relations (3·4) and (3·5) in equation (3·9) we obtain

$$\bar{s}(n) = \bar{f}(n)\ e^{\alpha\, \mu_n^2\, T} - \int_0^T \psi(t')\ e^{\alpha\, \mu_n^2 t'}\, dt'. \tag{3·10}$$

Applying the inversion transform (2·3), we obtain

$$s(r) = \sum_n \frac{1}{c_n}\left[\bar{f}(n)\ e^{\alpha\, \mu_n^2\, T} - \int_0^T \psi(t')\ e^{\alpha\, \mu_n^2\, t'}\, dt'\right] s_0(k_1, k_2, \mu_n r) \tag{3·11}$$

where c_n are given by (2·4) with $p = 0$ and $\psi(t')$ is obviously a known function given by (3·7)

4. Heat flow in a truncated wedge of finite height.

We shall begin by considering the distribution of temperature in the truncated wedge of finite height $h (a \leq r \leq b, 0 \leqslant \theta \leqslant \theta_0)$.
In this case the differential equation is

$$\frac{\partial u}{\partial t} = \alpha\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}\right) \tag{4·1}$$

$$a < r < b,\ 0 < \theta < \theta_0,\ o < z < h,\ 0 < t,$$

where α is diffusivity (constant)

with

(4·2) As $z \to 0$, $u \to 0$, for $a < r < b$, $0 < \theta < \theta_0$, $0 < t$,

(4·3) As $z \to h$, $u \to 0$, for $a < r < b$, $0 < \theta < \theta_0$, $0 < t$,

(4·4) As $r \to a$, $u \to 0$, for $0 < z < h$, $0 < \theta < \theta_0$, $0 < t$,

(4·5) As $r \to b$, $u \to \mu \cos n\theta$, for $0 < z < h$, $0 < \theta < \theta_0$, $0 < t$,

where n is a non negative integer and μ is a constant.

(4·6) As $t \to T$, $u \to u(r, \theta, z, T)$ (known), for $a < r < b$, $0 < z < h$, $0 < \theta < \theta_0$.

(4·7) As $t \to 0$, $u \to u(r, \theta, z, 0)$ (unknown), for $a < r < b$, $0 < z < h$, $0 < \theta < \theta_0$.

Now put

(4·8) $u = v_n(r, z, t) \cos n\theta$ then (4·1) to (4·7) becomes

(4·9) $$\frac{\partial v_n}{\partial t} = \alpha\left[\frac{\partial^2 v_n}{\partial r^2} + \frac{1}{r}\frac{\partial v_n}{\partial r} + \frac{\partial^2 v_n}{\partial z^2} - \frac{n^2}{r^2} v_n\right], \quad a < r < b,\ 0 < z < h,\ 0 < t$$

with

(4·10) As $z \to 0$, $v_n \to 0$, for $a < r < b$, $0 < t$.

(4·11) As $z \to h$, $v_n \to 0$, for $a < r < b$, $0 < t$.

(4·12) As $r \to a$, $v_n \to 0$, for $0 < z < h$, $0 < t$.

(4·13) As $r \to b$, $v_n \to \mu$, for $0 < z < h$, $0 < t$.

(4·14) As $t \to T$, $v_n \to v_n(r, z, T)$ (known), for $a < r < b$, $0 < z < h$.

(4·15) As $t \to 0$, $v_n \to v_n(r, z, 0)$ (unknown), for $a < r < b$, $0 < z < h$.

Applying Sine transform with respect to z defined in (2·7) to the equations (4·9) to (4·15), we obtain

(4·16) $$\frac{\partial \bar{v}_n}{\partial t} = \alpha\left[\frac{\partial^2 \bar{v}_n}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{v}_n}{\partial r} - \left(\frac{n^2}{r^2} + \frac{m^2\pi^2}{h^2}\right)\bar{v}_n\right], \quad a < r < b,\ 0 < t$$

where $$\bar{v}_n = \int_0^h v_n(r, z, t) \sin \frac{m\pi z}{h}\, dz.$$

with

(4·17) As $r \to a$, $\bar{v}_n \to 0$, for $0 < t$.

(4·18) As $r \to b$, $\bar{v}_n \to \frac{\mu h}{m\pi}(1 - \cos m\pi) = M$ (say), for $0 < t$.

(4·19) As $t \to T$, $\bar{v}_n \to \bar{v}_n(r, m, T)$ (known), for $a < r < b$.

(4·20) As $t \to 0$, $\bar{v}_n \to \bar{v}_n(r, m, 0)$ (unknown), for $a < r < b$.

By application to (4·16) of the transformation defined in (2·9), with respect to r, we obtain, using [7, p. 91, (6·58)]

$$\frac{d\bar{V}_n'}{dt} = \frac{2M\alpha}{\pi}\frac{J_n(\eta a)}{J_n(\eta b)} - \alpha\left(\eta^2 + \frac{m^2\pi^2}{h^2}\right)\bar{V}_n' \tag{4·21}$$

where
$$\bar{V}_n' = \int_a^b r\,\bar{V}_n(r, m, t)\,B_n(\eta r)\,dr$$

and we take η to be a positive root of the equation (2·10), with

$$\text{As } t \to T,\ \bar{V}_n' \to \bar{V}_n'(\eta, m, T) \text{ (known)}, \tag{4·22}$$

$$\text{As } t \to 0,\ \bar{V}_n' \to \bar{V}_n'(\eta, m, 0) \text{ (unknown)}. \tag{2·23}$$

The solution of (4·21) is

$$\bar{V}_n' = \frac{2\alpha M}{\pi q}\frac{J_n(\eta a)}{J_n(\eta b)} + A\,e^{-qt}, \tag{4·24}$$

where A is an arbitrary constant and

$$q = \alpha\left(\eta^2 + \frac{\pi^2 m^2}{h^2}\right) \tag{4·25}$$

Using the relations (4·22) and (4·23) in (4·24) we obtain

$$\bar{V}_n'(\eta, m, 0) = \bar{V}_n'(\eta, m, T)\,e^{qT} + \frac{2\alpha M J_n(\eta a)}{q\pi J_n(\eta b)}(1 - e^{qT}) \tag{4·26}$$

Applyir g the inversion formula (2·11) and then (2·8), we obtain

$$v_n(r, z, 0) = \frac{\pi^2}{h}\sum_{m=1}^{\infty}\sum_{\eta}\frac{\eta^2 J_n^2(\eta b)\,B_n(\eta r)}{[J_n^2(\eta a) - J_n^2(\eta b)]}\left[\bar{V}_n'(\eta, m, T)\,e^{qT} + \frac{2\alpha M J_n(\eta a)}{q\pi J_n(\eta b)}(1 - e^{qT})\right]\sin\frac{m\pi z}{h}.$$

And finally with the help of (4·8), we obtain

$$u(r, z, 0) = \sum_{n=0}^{\infty} v_n(r, z, 0)\cos n\theta \tag{4·27}$$

$$= \frac{\pi^2}{h}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{\eta}\frac{\eta^2 J_n^2(\eta b)\,B_n(\eta r)}{[J_n^2(\eta a) - J_n^2(\eta b)]}\left[\bar{V}_n'(\eta, m, T)\,e^{qT} + \frac{2\alpha\mu h(1 - \cos m\pi)}{q m \pi^2 J_n(\eta b)}(1 - e^{qT})\right]\sin\frac{m\pi z}{h}\cos n\theta,$$

where q is given by (4·25).

5. Heat flow in a semi-infinite solid containing an exterior plane crack with a circular boundary and an infinitely long cylindrical cavity.

In an attempt to construct the solution of time reversal problem in the mathematical theory of conduction related to axisymmetric temperature distribution of a semi-infinite solid containing an exterior plane crack with a circular boundary and an infinitely long cylindrical cavity, the axis of the cylinder being normal to the plane of the crack and passing through the centre of the circle bounding the crack, the following differential equation is encountered

$$\frac{\partial u}{\partial t} = \alpha\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2}\right), 1 < r < \infty, 0 < z < \infty, 0 < t \tag{5·1}$$

where α is the diffusivity, subject to the conditions

$$\text{As } r \to 1,\ u \to 0, \text{ for } 0 < z < \infty,\ 0 < t. \tag{5·2}$$

$$\text{As } r \to \infty,\ u \to 0, \text{ for } 0 < z < \infty,\ 0 < t. \tag{5·3}$$

$$\text{As } r \to \infty,\ \frac{\partial u}{\partial r} = o\left(r^{-\frac{1}{2}}\right), \text{ for } 0 < z < \infty,\ 0 < t. \tag{5·4}$$

$$\text{As } z \to 0,\ u \to 0, \text{ for } 1 < r < \infty,\ 0 < t. \tag{5·5}$$

$$\text{As } t \to T,\ u \to g(r, z), \text{ (known), for } 1 < r < \infty, 0 < z < \infty. \tag{5·6}$$

$$\text{As } t \to 0,\ u \to f(r, z), \text{ (unknown), for } 1 < r < \infty, 0 < z < \infty. \tag{5·7}$$

If we take the Weber-transform (2·12) of the equation (5·1) and use the well known properties of the Bessel functions and equations (5·2) to (5·4), we obtain

$$\frac{\partial u_w}{\partial t} = \alpha\left[-s^2 u_w + \frac{\partial^2 u_w}{\partial z^2}\right], 0 < z < \infty, 0 < t \tag{5·8}$$

where

$$u_w = \int_1^\infty r\, u(r, z, t)\left[J_0(rs)\, Y_0(s) - Y_0(rs)\, J_0(s)\right] dr$$

subject to the conditions

$$\text{As } z \to 0,\ u_w \to 0, \text{ for } 0 < t. \tag{5·9}$$

$$\text{As } t \to T,\ u_w \to g_w(s, z) \text{ (known), for } 0 < z < \infty. \tag{5·10}$$

$$\text{As } t \to 0,\ u_w \to f_w(s, z) \text{ (unknown), for } 0 < z < \infty. \tag{5·11}$$

Applying the Fourier Sine transform with respect to z defined in (2·14) to the equation (5·8) to (5·11), we obtain

$$\frac{du_w'}{dt} = -\alpha\,(s^2 + \xi^2)\, u_w', \quad 0 < t \tag{5·12}$$

where

$$u_w' = \int_0^\infty u_w(s, z, t) \sin(\xi z)\, dz$$

with

$$\text{As } t \to T,\ u_w' \to g_w'(s, \xi) \text{ (known).} \tag{5·13}$$

$$\text{As } t \to 0,\ u_w' \to f_w'(s, \xi) \text{ (unknown)} \tag{5·14}$$

The solution of (5·12) is

$$u_w' = A\, e^{-\alpha(s^2 + \xi^2)\, t} \tag{5·15}$$

where A is an arbitrary constant. Using the relations (5·13) and (5·14) in equation (5·15), we obtain

$$f_w'(s, \xi) = g_w'(s, \xi)\, e^{\alpha(s^2 + \xi^2)\, T} . \tag{5·16}$$

Applying the inverse transform (2·15) and then (2·13) to the equation (5·16), we obtain

$$f(r, z) = \frac{2}{\pi}\int_0^\infty s\, \frac{[J_0(r s)\, Y_0(s) - Y_0(r s)\, J_0(s)]}{[J_0^2(s) + Y_0^2(s)]} \times \int_0^\infty g_w'(s, \xi)\, e^{\alpha(s^2 + \xi^2)\, T} \sin(\xi z)\, d\xi\, ds. \tag{5·17}$$

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References

1. Marchi, E. and Zgrablich, G. *Proc. Edinburgh Math. Soc.*, **14**(2), 159–164, 1964/65.
2. Masket, A. V. *Amer. J. Phys.*, **33**(3), 196–201, 1965.
3. Roberts, J. K. and Miller, A. R. Heat and Thermodynamics. *Blackie and Son Limited, London*, 1951.
4. Sabherwal, K. C. *Indian J. Pure Appl. Phys.*, **3**, 449-450, 1965.
5. Sneddon, I. N. Fourier Transforms. *McGraw Hill Book Co. Inc., New York*, 1951.
6. Titchmarsh, E. C. Eigenfunction Expansions Associated with second order differential Equations. *Oxford University Press*, 1946.
7. Tranter, C. J. Integral Transforms in Mathematical Physics. *Methuen and Co. Ltd., Loudon.*

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Algebraic Classification of the Curvature Tensor in General Theory of Relativity

By

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Abstract

An algebraic classification of the curvature tensor in the general theory of Relativity has been studied in this paper, and a set of covariant tensorial conditions have been worked out which characterize the different classes. Some of the known line-elements have been tested in the classification and it has been found that the class III*a* and III*b* of the present classification correspand to Pure gravitational radiation. A comparative study of this classification has been made with that of Petrov's-classification and Roy-Radhakrishna's Self-conjugate Gravitational field.

1. Introduction. We shall assume in the following that the metric

$$ds^2 = g_{ij}\, dx^i\, dx^j$$

of the space-tim V_4 is reducible at a point to the Galelean-form,

$$ds^2 = -(dx^1)^2 - (dx^2)^2 - (dx^3)^2 + (dx^4)^2$$

i.e. we are dealing with the pseudo-euclidean metric at a point.

Any skew-symmetric tensor $A_{ij} \ldots = -A_{ji} \ldots$ defines a conjugate-tensor, $\overset{*}{A}_{ij} \ldots$ given by,

$$\overset{*}{A}_{ij} = \tfrac{1}{2}\sqrt{(-g)}\, \varepsilon_{ijhl}\, A^{kl}, \tag{1}$$

where

$$A^{kl} \ldots = g^{km}\, g^{ln}\, A_{mn} \tag{2}$$

Thus, we can define two types of conjugate-tensors from the 4-dimensional curvature-tensors R_{hijk}, as R_{hijk} is skew-symmetric in the pair of indices h, i and j, k, *viz* :

$$\overset{*}{R}_{hijk} = \tfrac{1}{2}\sqrt{(-g)}\ \varepsilon_{himn}\, R^{mn}{}_{jk} \tag{3}$$

and

$$\overset{**}{R}_{hijk} = \tfrac{1}{4}\,(-g)\, \varepsilon_{himn}\, \varepsilon_{jkpq}\, R^{mnpq} \tag{4}$$

We call $\overset{*}{R}_{hijk}$ and $\overset{**}{R}_{hijk}$ as the conjugate-tensors of the first and second kind respectively.

The twenty components of the tensors R_{hijk} can be divided into three groups :

$$(i)\quad R_{\alpha 4\beta 4},$$
$$(ii)\quad R_{\alpha\beta\gamma 4},$$
and $$(iii)\quad R\alpha\beta_{\gamma\delta}.$$

Where the Greek-indices $\alpha, \beta, \gamma, \ldots\ldots$ etc., take values 1, 2, 3 only while the Latin-indices $i, j, k, \ldots\ldots$, etc. take values 1, 2, 3, 4.

Then, it is clear that the sets (*ii*) and (*iii*) are equivalent to the sets $\overset{*}{R}_{\alpha 4\beta 4}$ and $\overset{**}{R}_{\alpha 4\beta 4}$ respectively, at the origin of normal coordinate-system [2].

The quantities $R_{\alpha 4\beta 4}$, $\overset{*}{R}_{\alpha 4\beta 4}$, and $\overset{**}{R}_{\alpha 4\beta 4}$ are tensors of the second-rank with respect to spatial transformations. Thus, in the 3–space x^4 = constant, the gravitational-field decomposes into three 3–tensor–fields of second-rank.

As has been done in [2], we denote the three tensors $R_{\alpha 4\beta 4}$, $\overset{*}{R}_{\alpha 4\beta 4}$ and $\overset{**}{R}_{\alpha 4\beta 4}$ respectively by $X_{\alpha\beta}$, $\Upsilon_{\alpha\beta}$ and $Z_{\alpha\beta}$ where $X_{\alpha\beta}$ and $Z_{\alpha\beta}$ are symmetric.

Then clearly
$$\Upsilon_{11} + \Upsilon_{22} + \Upsilon_{33} = 0, \tag{5}$$

If $R_{ij} = 0$, then
$$X_{\alpha\beta} + Z_{\alpha\beta} = 0, \tag{6}$$
$$\Upsilon_{\alpha\beta} - \Upsilon_{\beta\alpha} = 0. \tag{7}$$
and
$$X_{11} + X_{22} + X_{33} = 0. \tag{8}$$

Obviously, (7) implies the symmetry of $\Upsilon_{\alpha\beta}$,

Now, if λ_α be the eigen-vector of $X_{\alpha\beta}$, we have : $(X\alpha\beta - K\delta\alpha\beta)\,\lambda^\alpha = 0$, (9)
where K is the eigen-value corresponding to the victor λ^α.

But, this homogeneous equation has a non-trivial solution if

$$\text{Det. } (X_{\alpha\beta} - K\delta_{\alpha\beta}) = 0 \tag{10}$$

i.e. $$K^3 - \tfrac{1}{2} K X_{\alpha\beta} - X^{\alpha\beta} - \tfrac{1}{3} X_{\alpha\beta} X^{\beta\gamma} X_\gamma{}^\alpha = 0 \tag{11}$$

Now, we introduce the symmetric-complex tensor $W_{\alpha\beta}$ ([1] and [5]) defined as,

$$W_{\alpha\beta} = X_{\alpha\beta} + i\Upsilon_{\alpha\beta} \tag{12}$$

From this it follows that four-dimensional-transformations (Lorentz-Transformations) of the tensor R_{iklm} are equivalent to three-dimensional Complex-rotations applied to the tensor $W_{\alpha\beta}$. With respect to these rotations we can define eigen-values $K = K' + iK''$ and corresponding eigen-vectors λ_α (complex in general) as solutions of the system of equations,

$$K^3 - \tfrac{1}{2} K W_{\alpha\beta} W^{\alpha\beta} - \tfrac{1}{3} W_{\alpha\beta} W^{\beta\gamma} W_{\gamma}{}^{\alpha} = 0 \qquad (13)$$

and $$(W_{\alpha\beta} - K\delta_{\alpha\beta})\,\lambda^{\alpha} = 0 \qquad (14)$$

respectivety.

Obviously, (13) implies that the sum of the three roots is zero, *viz* :

$$K_1 + K_2 + K_3 = 0 \qquad (15)$$

II. Classification of R_{hijk} :

Depending on the number of independent eigen-values K_α of $W_{\alpha\beta}$, we arrive at the following classification :

Case (I). Let all the three eigen-values of $W_{\alpha\beta}$ *i.e.* the roots of (13) be different, then there exist $K_\alpha (= a_\alpha + ib_\alpha)$, $\alpha = 1, 2, 3$ satisfying (15). And the matrix for $W_{\alpha\beta}$ then reduces to the form.

$$(W_{\alpha\beta}) \equiv \begin{bmatrix} a_1 + ib_1 & 0 & 0 \\ 0 & a_2 + ib_2 & 0 \\ 0 & 0 & a_3 + ib_3 \end{bmatrix} \qquad (16)$$

Case (II). Let two out of three eigen-values of $W_{\alpha\beta}$ be equal, *i.e.* K_1, K_1 and K_2 $(= -2K_1)$ be the roots of the equation (13). Then we have the following two cases :

Case (IIa). Let X_α be the eigen-vector corresponding to the eigen-values $K_\alpha (= a_\alpha + ib_\alpha)$, $\alpha = 1, 1, 2$. Then, it is easy to show that the matrix for $W_{\alpha\beta}$ takes the form

$$(W_{\alpha\beta}) \equiv \begin{bmatrix} a_1 + ib_1 & 0 & 0 \\ 0 & a_1 + ib_1 & 0 \\ 0 & 0 & -2(a_1 + ib_1) \end{bmatrix} \qquad (17)$$

This form is achieved when we consider first one of the repeated eigen-value K_1 and then considering a 2×2 matrix with two distinct eigen-values K_1 and $-2K_1$.

Case (IIb). Let K_2 be the non-repeated eigen-value and X_2 be corresponding eigen-vector.

Now, if we consider first the non-repeated eigen-value and then a 2×2 matrix with two repeated eigen-values K_1. Then the canonical form for the matrix $W_{\alpha\beta}$ becomes,

$$(W_{\alpha\beta}) \equiv \begin{bmatrix} 2(a-ib) & 0 & 0 \\ 0 & -(a+d) + i\,(b-c) & -c-id \\ 0 & -c-id & -(a-d) + i\,(b+c) \end{bmatrix} \qquad (18)$$

Where $K_2 = 2(a-ib)$ and the rest repeated eigen-values are given by the 2×2 matrix, obtained by deleting first row and first-column.

Case (III). Let all the three eigen-values be equal, then (13) implies that $K = 0$, is the only solution.

There exist then only two possibilities for the matrix $W_{\alpha\beta}$, *viz.* :

$$(a) \quad \begin{bmatrix} 0 & A & B \\ A & 0 & C \\ B & C & 0 \end{bmatrix} \qquad \text{and } (b) \quad \begin{bmatrix} 0 & C & D \\ C & A & B \\ D & B & -A \end{bmatrix}$$

Case (IIIa). By hypothesis,

$$-\lambda^3 + \lambda(A^2 + B^2 + C^2) + 2ABC = 0,$$

must have three equal-roots. Now, considering different possibilities and on putting,

$$A = -b + ia\,;\, B = -(a + ib)$$
$$\text{and } C = 0.$$

we find that the form for the matrix of $W_{\alpha\beta}$ is,

$$\begin{bmatrix} 0 & -(a+ib) & -b+ia \\ -(a+ib) & 0 & 0 \\ -b+ia & 0 & 0 \end{bmatrix} \tag{19}$$

Case (IIIb). Similary, we find that the equation,

$$\lambda^3 - \lambda\,(A^2 + B^2 + C^2 + D^2) + AC^2 - AD^2) + 2BCD = 0,$$

must have three equal roots.

Now, considering different possibilities and substituting

$$A = -iB = -(a + ib) \text{ and } C = D = 0.$$

We get the form, for the matrix of $W_{\alpha\beta}$ as follows :

$$(W_{\alpha\beta}) \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & a+ib & -b+ia \\ 0 & -b+ia & -a-ib \end{bmatrix} \tag{20}$$

III. Covariant-study of the Case II and III :

In the following we study a set of covariant conditions (*cf* [5]) for the cases II and III.

(*a*) Let, there exist two distinct null-vectors say $\vec{l} = (1, 0, 0, 1)$ and $\vec{l} = (-1, 0, 0, 1)$ satisfying the relations,

$$R_{hijk}\, l^h\, l^j = 2a\, l_i\, l_k \text{ and } R^{*}_{hijk}\, l^h\, l^j = 2b\, l_i\, l_k \tag{21}$$

In this case we find that the only non-vanishing components of $X_{\alpha\beta}$ and $Y_{\alpha\beta}$ are :

$$\left.\begin{aligned} X_{22} &= -a = X_{33} \\ Y_{22} &= -b = Y_{33} \end{aligned}\right\} \tag{22}$$

Therefore, the matrix for $W_{\alpha\beta} = X_{\alpha\beta} + iY_{\alpha\beta}$, becomes

$$\begin{bmatrix} 2(a+ib) & 0 & 0 \\ 0 & -(a+ib) & 0 \\ 0 & 0 & -(a+ib) \end{bmatrix}$$

which according to the present classification is of the form (II*a*)

Thus, a necessary and sufficient condition in order that K_{hijk} may belong to the class (IIa) of the present classification is that there exist two null-vectors satisfying the relations (21).

(*b*) Let, there exist only one null-vector $\vec{l} = (1, 0, 0, 1)$ (or $\vec{l} = (-1, 0, 0, 1)$) satisfying the relations (21). Then, the results are :

$$\left.\begin{aligned} &X_{11} = 2a,\ Y_{11} = 2b,\ X_{13} = Y_{12},\ X_{12} = -Y_{13} \\ &2X_{23} = Y_{22} - Y_{33} \text{ and } 2Y_{23} = X_{33} - X_{22} \end{aligned}\right\} \tag{23}$$

Putting $$Y_{23} = c,\ X_{23} = -d,\ X_{13} = Y_{12} = p,\ X_{12} = -Y_{13} = q \tag{24}$$

It is easy to get from (23) and (24) that the matrix of $W_{\alpha\beta}$ is of the form,

$$(W_{\alpha\beta}) \equiv \begin{bmatrix} 2(a-ib) & q+ip & p-iq \\ q+ip & -(a-ib)-(c+id) & -d+ic \\ p-iq & -d+ic & -(a-ib)+(c+id) \end{bmatrix}$$

It is easy to see that this matrix has two equal eigen-values each equal to $-(a-ib)$ and a third one equal to $2(a-ib)$. And, hence this case is equivalent to the class (II*b*) of the present classification.

Thus, a necessary and sufficient condition in order that R_{hijk} may belong to the case (IIb) of the present classification is that there exist one null-vector satisfying the relations (21).

(*c*) Let us now consider the case when there exist a null-vector $\vec{l} = (1, 0, 0, 1)$ (or $\vec{l} = (1, 0, 0, 1)$) satisfying the relations,

$$R_{hijk}\, l^h\, l^j = 0 \text{ and } \overset{*}{R}_{hijk}\, l^h\, l^j = 0 \tag{25}$$

Then, clearly the results will be the same as in (24) except that $a = 0$ and $b = 0$ *i.e.* All the three eigen-values of the matrix $(W_{\alpha\beta})$ are zero and in this case the matrix reduces to that of the class III*a*.

Thus, a necessary and sufficient condition in order that R_{hijk} may belong to the class (IIIa) of the present classification is that there exist one null-vector satisfying the retation (25).

(*d*) Lastly, consider the case when there exist a null-vector $\vec{l} = (1, 0, 0, 1)$ (or $\vec{l} = (-1, 0, 0, 1)$ satisfying the relations,

$$R_{hijk}\, l^h = 0 \text{ and } \overset{*}{R}_{hijk}\, l^h = 0 \tag{26}$$

The relations (26) are the Lichnerowicz conditions for Pure gravitational radiation[3]).

In this case we find that,

$$\left.\begin{aligned} &X_{11} = X_{12} = X_{13} = Y_{11} = Y_{12} = Y_{13} = 0 \\ &Y_{23} = X_{22} = X_{33} \text{ and } X_{23} = Y_{22} = Y_{33} \end{aligned}\right\} \tag{27}$$

Let us put,

$$Y_{23} = -X_{22} = X_{33} = p \text{ and } X_{23} = Y_{22} = -Y_{33} = q \tag{28}$$

Then the matrix of $W_{\alpha\beta}$ takes the form,

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -p+iq & q+ip \\ 0 & q+ip & p-iq \end{bmatrix} \tag{29}$$

Which is similar to the case (III*b*) of the present classification. It should be noted that the same results are obtained when either of the two null-vector is used.

Thus, a necessary and sufficient condition in order that R_{hijk} may belong to the class (IIIb) of the present classification is that there exist a null-vector satisfying the relations (26).

IV. Study of some of the line-elements :

(*i*) *Takeno's plane wave solution*[4] :

Takeno[4] has studied the following line-element,

$$ds^2 = -A\,dx^2 - 2D\,dxdy - B\,dy^2 - dz^2 + dt^2 \tag{30}$$

for which,

$$\left.\begin{aligned} &R_{1313} = -R_{1314} = R_{1414} = U,\ R_{2323} = -R_{2324} = R_{2424} = v \\ &\text{and } R_{1323} = -R_{1324} = -R_{1423} = R_{1424} = w \end{aligned}\right\} \tag{31}$$

It is easy to see that for the line-element (35), Lichnerowiczs condition (26) are satisfied by taking the null-vector $\vec{l} = (0, 0, 1, 1)$.

Hence, the Takeno's line-element belongs to the class (III*b*) of the present classification which therefore corresponds to Pure gravitational radiation.

(*ii*) *Schwarzschild's – Exterior solution* :

The line-element is,

$$ds^2 = -e^{\lambda}\, dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta\, d\phi^2 + = e^{\nu}\, dt^2 \qquad (32)$$

with $\nu = -\lambda$ and $e^{\nu} = (1 - 2m/r)$, (*cf.* [6]),

The non-vanishing components of R_{hijk} in this case are :

$$R_{1414} = -\tfrac{1}{4}\nu''\, e^{\nu} - \tfrac{1}{2}\nu'^2\, e^{\nu} + \tfrac{1}{4}\lambda'\nu' e^{\nu},\ R_{1212} = -\tfrac{1}{2} r\,\lambda',$$

$$R_{2323} = r^2(e^{-\lambda} - 1)\sin^2\theta,\ R_{1313} = -\tfrac{1}{2} r\lambda' \sin^2\theta,$$

$$R_{3434} = -\tfrac{1}{2} r\nu' \sin^2\theta . e^{\nu-\lambda},\ R_{2424} = -\tfrac{1}{2} r\nu'\, e^{\nu-\lambda}.$$

Now, it is easy to see that in this case by taking the null-vector

$$\vec{l} = (K^{-\frac{1}{2}}, 0, 0, K^{1/2}), \text{ where } K = (1 - 2m/r) \equiv e^{\nu}$$

the conditions for the class (II*b*) *viz.* the relations (21) are satisfied.

Hence, the Schwarzschilds-line element is of the class (II*b*) according to this classification.

(*iii*) *Einstein – Rosen metric* :

$$ds^2 = -e^{2\gamma-2\psi}(d\rho^2 - dt^2) - e^{2\psi}\, dz^2 - \rho^2 e^{-2\psi}\, d\phi^2 \qquad (33)$$

where $\gamma = \gamma(\rho, t)$ and $\psi = \psi(\rho, t)$.

By virtue of[3] we can say that the above Einstein - Rosen metric could be reduced to the form :

$$ds^2 = e^{2\phi}\,(dt^2 - dx^2) - U^2(e^{-2\beta}\, dy^2 + e^{2\beta}\, dz^2) \qquad (34)$$

where β, ϕ, and U are functions of x and t.

Now, it has already been shown by Lichnerowicz[3] that by a suitable transformations the Einstein-Rosen Metric is reducible to the form,

$$ds^2 = d\bar{u}\, d\bar{v} - d\bar{y}^2 + d\bar{z}^2) - 2\beta'(\bar{y}d\bar{y} - \bar{z}d\bar{z} - \frac{\bar{y}^2 - \bar{z}^2}{u}\, du)\, du - \beta'^2 \bar{u}\,\bar{v}\, du^2 \qquad (35)$$

in the region $U_0 \leqslant U \leqslant U_1$,

and reduces to,

$$ds^2 = d\bar{u}\, d\bar{v} - (d\bar{y}^2 + d\bar{z}^2), \qquad (36)$$

in the region $U \leqslant U_0$ and $U \geqslant U_1$

Where U_0 and U_1 are two positive numbers and β' is a definite-function of class C^1, C^3 by parts) in $U_0 \leqslant U \leqslant U_1$ such that

$$\beta'(U_0) = \beta'(U_1) = 0 \text{ and } \beta''(U_0) = \beta''(U) = 0.$$

Where, $\bar{u}, \bar{v}, \bar{y}, \bar{z}$, is a new coordinate system.

Further it has been shown in[3] that (35) satisfies the covariant conditions (26) and that (36) give an Euclidean space.

Hence in the region where it reduces to (35) the Einstein-Rosen metric belong to the class (III*b*) of Pure gravitational radiation.

V. Comparative study of the present classification with that of Petrov[1] and Roy-Radhakrishna[2] :

In the first case Petrov has taken 3-independent eigen-vectors such that their square are different from zero.

This case is obviously same as the class I of the present classification, as independent eigen-values imply independent eigen-vectors.

In the 2nd case of Petrov, when there are two independent eigen-vectors. The square of one of them is equal to zero. Then the matrices are :

$$X_{\alpha\beta} = \begin{bmatrix} \sigma' & \phi & 0 \\ \phi & \sigma' & 0 \\ 0 & 0 & -2\sigma' \end{bmatrix} \text{ and } Y_{\alpha\beta} = \begin{bmatrix} \sigma''-\phi & 0 & 0 \\ 0 & \sigma''+\phi & 0 \\ 0 & 0 & -2\sigma'' \end{bmatrix} \tag{37}$$

which tell us about the non-vanishing components of R_{hijk} at a point in an empty gravitational field.

It is easy to verify that the relations (21) which are the necessary and sufficient conditions for the class (II*b*) are satisfied by the curvature tensor given by (37) if we take the null vector to be $\vec{l} = (0, 0, 1, 1)$.

Hence, Petrov's 2nd case corresponds to the class (II*b*) of the present classification.

Lastly, take the 3rd case of Petrov when there is just one-independent eigen-vector and its square is equal to zero. The two matrices are,

$$X_{\alpha\beta} = \begin{bmatrix} 0 & 0 & \phi \\ 0 & 0 & 0 \\ \phi & 0 & 0 \end{bmatrix} \text{ and } Y_{\alpha\beta} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \phi \\ 0 & \phi & 0 \end{bmatrix} \tag{38}$$

which imply

$$R_{1434} = -R_{2312} = \phi \text{ and } R_{4131} = R_{4223} = \phi$$

Now, it is easy to verify that the conditions (25), which are the necessary and sufficient conditions for the class III*a* are satisfied, by the curvature-tensor given by (38) if we take the null-vector to be $\vec{l} = (0, 0, 1, 1)$.

Hence, the 3rd case of Petrov corresponds to the case III*a* of the present classification.

Coming to the comparision with that of Roy and Radhakrishna we find that the only apparent difference seems to be that Petrov has taken the rotation of the complex-tensor $W_{\alpha\beta}$ *i.e.* the rotation of $X_{\alpha\beta}$ and $Y_{\alpha\beta}$ simultaneously, where as Roy and Radhakrishna have taken rotations separately but through the same angle, amounting to the same thing.

We consider the result from[2] *viz.* "the necessary and sufficient condition that an empty gravitational-field be self conjugate of type (*A*), is that,

$$P_{jkmn} \equiv R^{h}{}_{ijk}\, R^{i}{}_{hmn} = 0 \tag{39}$$

And the gravitational field satisfying the condition (39) also possesses the property of being a field of pure-gravitational radiation"

According to[2] the matrices for $X_{\alpha\beta}$ and $Y_{\alpha\beta}$, with a specific orientation, are

$$(X_{\alpha\beta}) \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & -A \end{bmatrix} \quad \text{and} \quad (Y_{\alpha\beta}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & B \\ 0 & B & 0 \end{bmatrix} \tag{40}$$

Now it is easy to varify that the conditions (26) which are the necessary and sufficient for the class III*b*, are satisfied by the curvature-tensor given by (40) if we take the null-vector $\vec{l} = (1, 0, 0, 1)$.

Hence the "self conjugate gravitational field of type *A*" of[2] corresponds to the class (III*b*) of the present classification.

Thus, we see from the above comparison that the Petrov's classifications of type I, II and III corresponds to the class I, II(*b*) and III(*a*) respectively of the present classification. Moreover, the present classification seems to be more exhaustive as the class II(*a*) and III(*b*) have no correspondence in the Petrov's case. It is, however, found that the type III(*b*) correspondes to the "self conjugate gravitational-field of type *A*" of[2]. This clearly shows that the classes III(*a*) and (III*b*) of the present classification corresponds to the case of "Pure gravitational-radiation".

Further more, it has been possible to find out a set of covariant criteria which enables us to show easily as to which class a particular curvature tensor will belong.

References

1. Petrov, A. Z. *Sci. Not. de l'Universite de Kazan*, **114**, 55, 1954.
2. Roy, S. R. and Radha Krishna, L. Self conjugate Gravitational fields I. *Proc. of the Roy. Soc., A*, **275**, 245–256, 1963.
3. Lichnerowicz, A. Ondes et. Radiations Electromagnetiques et Gravitationelles En Relative Generale. *Annali di matematica pura ed applicata, Serie*, **4**, *Tomo L*, 1960.
4. Takeno, H. The Mathematical Theory of Plane Gravitational waves in General-Relativity, Scientific Reports of the Research Institute for Theoretical Physics, *Hiroshima University*, No 1, Chapter I, 21-23, 1961.
5. Bell, L. la Radiation Gravitationnelle These, Sorbone, 1960.
6. Tolman, R. C. Relativity, Thermodynamics and Cosmology. *Oxford Press*, Art, 82, 202–205, 1962.

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New modified method for the determination of true density of soils

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The true density of soils is a measure of their porosity which is an important factor in determining their use in cultivation, building foundation, oil drilling, etc. Therefore, it is evident that accurate measurement of true density is of great importance. However, its determination by the present method does not seem to be quite correct because the true volume of soil particles is measured by the displacement of water by soil. As the soil is a mixture of Organic and Inorganic compounds, the true volume of soil particles cannot be determined by the displacement of water because some of the Inorganic salt fractions will dissolve and, therefore, will not contribute to the displacement of water.

In the existing literature, true density measurements are reported without considering the volume of dissolved fractions of the soil. This present communication reports a modified method of determining the true density of soils by taking into account the dissolved inorganic salts in water. The physical property of an Organic solvent such as kerosene oil in which inorganic salts do not dissolve, has been utilized in this determination. For this purpose, first soil is placed in water for the displacement measurement and then displacement of the dissolved fractions in water is determined in organic solvent such as kerosene oil. The addition of these two displacements of the two liquids, gives the total volume of soil particles.

Experimental

A known amount, w. gm., of 2 m.m. sieved oven dry soil was taken in measuring flasks of 50 or 100 ml capacity. The soil was equilibrated with distilled water and the total volume made upto the mark. On weighing the weight of the soil and distilled water filled in the flask upto the mark was found to be W_1 gm. This mixture of the soil and distilled water was filtered and filtrate collected. The measuring flask was cleaned and filled upto the same mark with distilled water. The weight of the distilled water was found to be W_2 gm.

The above filtrate was dried and x gm of residue was obtained. The volume of this residue was determined by displacement of the kerosene oil. The experimental procedure is the same as described above. It can be easily shown that the true density of the soil particles will be given by the following equation.

$$D = \frac{w d_1 d_2}{d_2 (W_2 + w - W_1) + d_1 (X_2 + x - X_1)}$$

where d_1 and d_2 = density of the distilled water and kerosene oil respectively.

TABLE

True density of various soils by old and new method at 25° ± 1°C of various fields of Doon Valley

S. No.	Soils of	Wt. of 2 mm sieved oven dry soil taken in grams	Wt. of distilled water displaced by water insoluble part of soil	Wt. of distilled kerosene oil displaced by water soluble part of soil	Volume occupied by the soil water insoluble part in water	Volume occupied by water soluble part of soil in kerosene oil	Total true volume of soil particles	True density determined by old routine method	True density determined by new modified method
1.	Rajpur Road Area	29·5700	11·1924	0·0252	11·2452	0·0321	11·2773	2·6295	2·6223
2.	Sahastradhara road	29·6720	11·3108	0·0210	11·3642	0·0267	11 3909	2·6110	2·6048
3.	Savella field	29·6050	11·0134	0·104	11·6720	0·0132	11·6652	2·5407	2·5387
4.	Majhon field	29·6026	11·0882	0·0134	11·1405	0·0170	11·1575	2·6572	2·6531
5.	Harwanswala Tea Estate	29·6480	11·1105	0·0160	11·1934	0·0203	11·2137	2·6487	2·6439
6.	Ambiwala Tea Estate	29·5380	10·8300	0·0150	10·8811	0·0191	10·9002	2·7137	2·7092
7.	Balupur field	29·5940	10·8766	0·0180	10·9229	0·0229	10·9508	2·7081	2·7024
8.	Niranjanpur Tea Estate (At Saharanpur Road)	29·6842	11·1442	0·0150	11·6969	0·0191	11·2160	2·6511	2·6465
9.	Niranjanpur Leechee garden (at Saharanpur Road)	29·5374	10·7255	0·0256	10·7761	0·0326	10·8087	2·7410	2·7327
10.	Dalanwala Area	29·4646	11·0699	0·0234	11·2327	0·0298	11·2625	2·6231	2·6161
11.	Niranjanpur Tea Estate (near FRI)	29·6288	11·0068	0·0126	11·1592	0·0160	11·1752	2·6551	2·6513
12.	F. R. I. Area	29·5884	10·9823	0·0100	11·6342	0·0127	11·0479	2·6815	2·6781
13.	College fields soils	29·5750	10·7756	0·0120	10·8264	0·0152	10·8147	2·7317	2·7278
14.	Kishanpur Area	29·6046	10·8412	0·0146	10·8923	0·0186	10·9109	2·7179	2·7133

X_1 = weight of the residue and the kerosene oil

X_2 = weight of kerosene oil having the same volume, as that of the mixture of residue and kerosene oil.

Result

The true density of the soil calculated by the old method and the present modified method are given in the table.

It is evident from the table that true density of various soils determined by the new modified method is less than that determined by the old method.

In Doon Valley areas, according to experiments conducted,

(*a*) the soils No. 1 and No. 10, having densities 2·6223 and 2·6161 are best for the production of Leechee.

(*b*) the soil No. 3 having density 2·5387 is most suitable for rice cultivation, and

(*c*) the soil No. 6 having density 2·7092 is ideal for tea production.

Further work on these soils regarding cultivation of Basmati Rice, Tea and Leechee is in progress.

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where I_n is the Bessel function of the first kind, I_n is the modified Bessel function of the first kind, and $\psi_n(\xi)$ and $X_n(\xi)$ are arbitrary functions depending on the boundary conditions of the problem.

Applying the condition (2·2) and (2·3) to the relation (2·5), we obtain the following pair of dual integral equations :

$$\int_0^\infty \psi_n(\xi) J_n(\rho\xi)\, d\xi + \int_0^\infty \chi_n(\xi) I_n(\rho\xi)\, d\xi = f_n(\rho), \quad (0 < \rho < 1) \tag{2·6}$$

$$\int_0^\infty \xi^{-1} \psi_n(\xi) J_n(\rho\xi)\, d\xi = 0, \ (1 < \rho < c), \tag{2·7}$$

and the condition (2·4) gives

$$\int_0^\infty \xi^{-1} \psi_n(\xi) J_n(c\xi)\, e^{-z\xi}\, d\xi - \int_0^\infty \xi^{-1} \chi_n(\xi) I_n(c\xi) \sin z\xi\, d\xi = h(z), \ z > 0 \tag{2·8}$$

To solve the equations (2·7) and (2·8), we assume

$$\int_0^\infty \xi^{-1} \psi_n(\xi) J_n(\rho\xi)\, d\xi = \begin{cases} \rho^n \int_\rho^1 \dfrac{g(t)\, dt}{\sqrt{(t^2 - \rho^2)}}, & (0 < \rho < 1), \\ 0, & (1 < \rho < c), \end{cases}$$

then on applying Hankel inversion theorem, we get

$$\psi_n(\xi) = \xi^2 \int_0^1 g(t) \int_0^t \frac{\rho^{n+1} J_n(\xi\rho)\, d\rho}{\sqrt{(t^2 - \rho^2)}}\, dt. \tag{2·9}$$

In the equation (2·9) the function $g(t)$ is yet unspecified.

On using the result [4, p. 26] the equation (2·9) reduces to

$$\psi_n(\xi) = \left(\frac{\pi}{2}\xi^3\right)^{\frac{1}{2}} \int_0^1 t^{n+\frac{1}{2}} g(t) J_{n+\frac{1}{2}}(\xi t)\, dt. \tag{2·10}$$

If we put the above value of $\psi_n(\xi)$ in the equation (2·7) and interchange the order of integration, we find that (2·7) is identically satisfied by $\psi_n(\xi)$ for all those functions $g(t)$ for which we can interchange the order of integration.

Now multiply (2·6) by ρ^{n+1} and integrate it w.r.t. ρ from 0 to ρ, we obtain

$$\int_0^\infty \xi^{-1} \psi_n(\xi) J_{n+1}(\rho\xi)\, d\xi + \int_0^\infty \xi^{-1} \chi_n(\xi) I_{n+1}(\rho\xi)\, d\xi = \rho^{-n-1} \int_0^\rho u^{n+1} f_n(u)\, du, \ (0 < \rho < 1). \tag{2·11}$$

If we now substitute the value of $\psi_n(\xi)$ from (2·10) in the equation (2·11), we obtain that it is reduced to the equation

$$\sqrt{(\pi/2)} \int_0^1 t^{n+\frac{1}{2}} g(t) \left(\int_0^\infty \xi^{\frac{1}{2}} J_{n+1}(\rho\xi) J_{n+\frac{1}{2}}(t\xi)\, d\xi \right) dt + \int_0^\infty I_{n+1}(\rho\xi) \xi^{-1} \chi_n(\xi)\, d\xi = \rho^{-n-1} \int_0^\rho u^{n+1} f_n(u)\, du, \ (0 < \rho < 1),$$

which on using the result [4, p. 48] reduces to

$$\int_0^\rho \frac{t^{2n+1} g(t)\, dt}{\sqrt{(\rho^2 - t^2)}} = \int_0^\rho u^{n+1} f_n(u)\, du - \rho^{n+1} \int_0^\infty \chi_n(\xi) I_{n+1}(\rho\xi)\, d\xi, \quad (0 < \rho < 1).$$

This is a standard Abel integral equation [7, p. 229] and its solution is given by

$$t^{2n+1}\, g(t) = \frac{2}{\pi}\,\frac{d}{dt}\int_0^{\rho} \frac{\rho}{\sqrt{(t^2-\rho^2)}} \left\{ \int_0^{\rho} u^{n+1} f_n(u)\, du \right\} d\rho$$

$$- \frac{2}{\pi}\,\frac{d}{dt}\int_0^{t} \frac{\rho^{n+2}}{\sqrt{(t^2-\rho^2)}} \left\{ \int_0^{\infty} \chi_n(\xi)\, I_{n+1}(\rho\xi)\, d\xi \right\} d\rho, \ (0 < t < 1). \qquad (2{\cdot}12)$$

The expression for $g(t)$ still involves an unknown function $\chi_n(\xi)$. This is determined as follows :

Taking the Fourier sine transform with respect to z of both sides of (2·8), we obtain

$$\xi^{-1}\, \chi_n(\xi)\, I_n(c\xi) = \sqrt{\left(\frac{2}{\pi}\right)}\, \xi \int_0^{\infty} \eta^{-1}\, \psi_n(\eta)\, J_n(c\eta)\, (\xi^2 + \eta^2)^{-1}\, d\eta - \sqrt{\left(\frac{2}{\pi}\right)} F_s(\xi),$$

where $F_s(\xi)$ denotes the Fourier sine transform

$$\sqrt{\left(\frac{2}{\pi}\right)} \int_0^{\infty} h(z) \sin(\xi z)\, dz,$$

of the function $h(z)$. Substituting the value of $\psi_n(\eta)$ from (2·10) in the above expression and using the result [4, p. 49], we find

$$\chi_n(\xi) = \xi^{3/2}\, \frac{K_n(c\xi)}{I_n(c\xi)} \int_0^{c} t^{n+\frac{1}{2}}\, g(t)\, I_{n+\frac{1}{2}}(t\xi)\, dt - \sqrt{\frac{2}{\pi}}\, \frac{F_s(\xi)}{I_n(c\xi)}. \qquad (2{\cdot}13)$$

If we now combine (2·12) and (2·13), we obtain the integral equation

$$t^{2n+1}\, g(t) = \frac{2}{\pi}\,\frac{d}{dt}\int_0^{t} \frac{\rho}{\sqrt{(t^2-\rho^2)}} \left\{ \int_0^{\infty} u^{n+1} f_n(u)\, du\, d\rho \right.$$

$$\left. - \frac{2}{\pi}\, t^{n+3/2} \int_0^{\infty} \xi^{\frac{1}{2}}\, \frac{F_s(\xi)}{I_n(c\xi)}\, I_{n+\frac{1}{2}}(t\xi)\, d\xi \right\}$$

$$- \int_0^{c} u^{n+\frac{1}{2}}\, g(u) \int_0^{\infty} \xi^{\frac{1}{2}}\, \frac{K_n(c\xi)}{I_n(c\xi)}\, I_{n+\frac{1}{2}}(u\xi)\, t^{n+3/2}\, I_{n+\frac{1}{2}}(\xi t)\, d\xi\, du,$$

or

$$g(t) + t^{-n+\frac{1}{2}} \int_0^{c} u^{n+\frac{1}{2}}\, g(u)\, K(t, u)\, du = \phi(t), \ (0 < t < 1), \qquad (2{\cdot}14)$$

where

$$K(t, u) = \frac{2}{\pi} \int_0^{\infty} \frac{K_n(c\xi)}{I_n(c\xi)}\, I_{n+\frac{1}{2}}(u\xi)\, I_{n+\frac{1}{2}}(t\xi)\, d\xi,$$

$$\phi(t) = \frac{2}{\pi}\, t^{-2n-1}\, \frac{d}{dt} \int_0^{t} \frac{\rho}{\sqrt{(t^2-\rho^2)}} \int_0^{\rho} u^{n+1} f_n(u)\, du\, d\rho$$

$$- \frac{2}{\pi} t^{-n+\frac{1}{2}} \int_0^{\infty} \xi^{\frac{1}{2}}\, \frac{F_s(\xi)}{I_n(c\xi)} I_{n+\frac{1}{2}}(t\xi)\, d\xi.$$

The equation (2·14) is a Fredholm integral equation of the second kind for determining $g(t)$. Once the function is known the solution of the pair of equations (2·6) and (2·7) follows immediately from the equation (2·10).

3. In the above problem, we find that along with the mixed type of conditions on the plane $z = 0$ the simple conditions on the lateral surface of the cavity is that it is maintained at potential $h(z) \cos n\theta$. Now we consider the problem when along with the same conditions as previously, on the plane $z = 0$, the surface density of charge on its lateral surface is $h(z) \cos n\theta$. Thus the boundary condition (2·4) is replaced by

$$\frac{\partial \phi(c, \theta, z)}{\partial \rho} = f_n(z) \cos n\theta, \ (0 < z < \infty). \tag{3·1}$$

As in the previous case $g(t)$ is determined from the equation (2·12). Here the only difference is that the function χ_n will be represented by a different integral because the boundary condition (2·4) has been replaced by the condition (3·1).

Applying the condition (3·1) to the equation (2·5), we obtain

$$\int_0^\infty \psi_n(\xi) J_n'(c\xi) e^{-z\xi} d\xi - \int_0^\infty \chi_n(\xi) I_n'(c\xi) \sin z\xi \, d\xi = h(z), \ (0<z<\infty), \tag{3·2}$$

which on taking Fourier sine transform reduces to

$$\chi_n(\xi) = \frac{2}{\pi} \frac{\xi}{I_n'(c\xi)} \int_0^\infty \psi_n(\eta) J_n'(c\eta) (\xi^2 + \eta^2)^{-1} d_\eta - \frac{2}{\pi} \frac{F_s(\xi)}{I_n'(c\xi)}$$

$$= \frac{\xi^{3/2} K_n'(c\xi)}{I_n'(c\xi)} \int_0^c t^{n+1/2} g(t) I_{n+1/2}(\xi t) dt - \frac{2}{\pi} \frac{F_s(\xi)}{I_n'(c\xi)}. \tag{3·3}$$

Substituting the value of $\chi_n(\xi)$ from (3·3) into (2·12), we obtain

$$g(t) + t^{-n+1/2} \int_0^c u^{n+1/2} g(u) K(t, u) du = \phi(t), \ (0 < t < 1), \tag{3·4}$$

where

$$K(t, u) = \frac{2}{\pi} \int_0^\infty \xi^2 \frac{K_n'(c\xi)}{I_n'(c\xi)} I_{n+\frac{1}{2}}(u\xi) I_{n+\frac{1}{2}}(t\xi) \, d\xi,$$

where $\phi(t)$ having the same value as before.

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References

1. Collins, W. D. Note on an electrified circular disk situated inside a coaxial infinite hollow cylinder. *Proc. Camb. Philos. Soc.*, **57**, 623–627, 1961.
2. Copson, E. T. On problem of electrified disk. *Proc. Edin. Math. Soc.*, **2**, 14-19, 1947-50.
3. ——— On certain dual integral equations. *Proc. Glasgow Math. Assoc.*, **5**, 21-24, 1961.
4. Erdelyi, A. (*ed*). Tables of Integral Transforms, *McGraw-Hill, New York*, **Vol. II**, 1954.
5. Sneddon, I. N. Note on an electrified disk situated inside a coaxial infinite hollow cylinder. *Proc. Camb. Philos. Soc.*, **58**, 621–624, 1962.
6. ——— The elementary solution of dual integral equations. *Proc. Glasgow Math. Assoc.*, **4**, 108-110, 1960.
7. Wittaker, E. T. and Watson, G. N. A course of Modern Analysis, *4th ed. Cambridge Universiy Press*, 1946.

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Utilization of an alkali soil by the addition of Nag-phana H. (Cactus) and Argemone-Mexicana

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Abstract

An alkali land treated with Cactus (Nag-phana H.) and gypsum was found to be an economic method of its utilization in terms of paddy and wheat grain yields. On an average, the combination of Nag-phana H. (Cactus) and superphosphate was also found to give good results but not better than the former. The results were compared with the alkali soil treated with Argemone-Mexicana, and with other common combinations of gypsum and superphosphate.

Introduction

India especially Uttar Pradesh and Punjab have a high percentage of alkali soils which are lying idle for their unfavourable phy ico-chemical effects on the growth and development of different crops. It is already well known that the addition of acids or acid producing substances like organic matter, sulphuric acid, sulphur, aluminium sulphate, ferrous sulphate and salts like gypsum and mono-calcium phosphate etc. have beneficial effects on the reclaimation of alkali soils (Leather 1914 ; Overstreet *et al.* 1951, Dalton *et al*, 1952 ; Dhar, 1955 ; Makarets, 1957). Most of these observations are along with such organic substances for the growth and development of which some attention of the cultivator or a good amount of expenditure is involved. In such studies, the use of weeds was little tried either singly or in combination with such important alkali reclaiming substances as gypsum and superphosphate.

In this paper, the results obtained in terms of paddy and wheat yields in an alkali soil in presence of two common weeds, Nag-phana H. (Cactus) and Argemone-Mexicana along with gypsum and superphosphate have been raported and discussed.

Experimental

Field trials were done in Allahabad in an area which is fairly uniform and has been under cultivation since a long time. Original soil samples were taken out from different layers and from different spots upto the depth of 8" in order to determine the uniformity of the soil of the selected field. Randomized samples, were therefore, taken out and chemically analysed by the usual methods of soil analysis (De, 1965). Nag-Phana H. and Argemone-Mexicana were also analysed for certain important constitutent (De, 1965).

Two lands were selected of the same field and 2 blocks were made of 1/2·2 of an acre in size in which Nag-phana H. (Cactus) and Argemone-Mexicana alone and along with other inorganic substances like superphosphate and gypsum were incorporated as treatments shown in Table 2 to determine the influence of addition

of Nag-phana H. (Cactus) and Argemone-Mexicana in crop production without composting. Sufficient time, water and ploughing (six times) were given for the complete oxidation and decomposition of organic matter added directly before the sowing of the crops. It was left for about 50 days and and then soil samples were collected and analysed to know the chemical status of the blocks. The preparation of field and growing of the paddy seedlings were in process during all this time.

After the full preparation of the field, paddy seedlings were transferred in the month of April and a successful crop was grown by the subsequent supply of irrigation water and weeding etc. The paddy crop was harvested in the month of June and the production was known in terms of grain and straw. Again the soil was analysed by the usual chemical methods of analysis (Tables 1 to 5). In the next block wheat was sown in the month of October and the crop was harvested in the month of March. Production was recorded and analysis of soil was made. In the next year one more crop of paddy was taken in the same fashion as noted above and the production in terms of grain and straw was observed.

Schemes of the Experiments

Crop taken	..	Paddy, wheat and again paddy.
Irrigation given	..	Paddy, 1st irrigation at the time of transplanting. 2nd after 15 days. 3rd after one month. 4th after one and half-month. Wheat – 1st irrigation after 20 days of sowing. 2nd after one month of the sowing. 3rd after $1\frac{1}{4}$ months of sowing.
Design of the experiment	..	Randomized block layout.
No. of treatments	..	9
No. of replications/Block	..	4
No. of plots/Block	..	36
Area and size of the plots	..	1/80th of an acre.

Rate of Added Substances

Organic matters as Nag-phana H (Cactus) and Argemone-Mexicana		5 tons/acre
Superphosphate	..	50 lbs (P_2O_5)/acre
Gypsum	..	5 tons/acre.

Treatments

1. Soil alone (control)
2. Soil + Gypsum.
3. Soil + Superphosphate
4. Soil + Nag-phana H. (Cactus)
5. Soil + Cactus + Gypsum
6. Soil + Cactus + Superphosphate.
7. Soil + Argemone-Mexicana
8. Soil + Argemone–Mexicana + Gypsum.
9. Soil + Argemone–Mexicana + Superphosphate.

Resplts and Discussion

A perusal of Table No. 1 will clearly show that Nag-phana H. (Cactus) contains a high percentage of calcium, phosphorus and nitrogen in comparison to Argemone-Mexicana. This weed on the other hand contains a higher percentage of carbon indicating a greater possibility of the release of carbonic acid after its decomposition. However, as is evident in Table No. 2 and 3, the combination No. 5 gave the best result both in terms of paddy and wheat yields and in diminishing water-soluble sodium and water-soluble calcium. This is expected as the good effects are not only due to the release of carbonic acid from the easy decomposition but also because of its high contents of calcium accentuated by the addition of calcium sulphate. Thus, in the soil treated with both Nag-phana H. (Cactus) and gypsum the exchange positions were taken more by calcium ions subsequently releasing sodium ions which as sodium sulphate and sodium hydroxide were removed in considerable extent with the addition of irrigation water. As the removal of NaOH (high electrical conductivity) is easier than Na_2SO_4, the electrical conductivity and pH of the resultant soil (Table 3) were also reduced. These appear to be the reason why soil treated with cactus and gypsum were comparatively lower in sodium and calcium and in general gave the best effects both in terms of paddy grain (highest yield 800 Kg./acre) and wheat grain (highest yield, 852 Kg/acre) yields as given in Table No. 6 and 7. In the end, it may be pointed out that the second crop of paddy (Table No. 8) taken in the same field gave a further increase in yield, the higher quantity being with the combination of 5 indicating clearly that although after 50 days of the addition of cactus (Nag-phana H.) and gypsum a good crop can be taken out of the soil but a better yield is still possible to obtain without further addition of either gypsum or Nag-phana H. as a result of greater reduction in the undesirable properties of the alkali soil used in the experiment.

TABLE 1

Chemical analysis of the alkali soil, Nag-phana H. and Argemone-Mexicana

Determining value	% in Alkali soils	% in Nag-phana H. (Cactus)	% in Argemone-Mexicana
Fe_2O_3	3·9845	0·2500	0·4014
Total CaO	1·1240	12·0452	0·8025
Exch. Ca++	5·40 m.e./100 gms.	-	-
Total MgO	1·0234	0·6023	0·5998
Total P_2O_5	0·1046	1·2034	0·5003
Total K_2O	0·7544	2·4548	0·8986
Total – C	0·302	28·9846	39·9843
Total – N	0·0435	1·8024	0·6698

pH = 9·0

Electrical conductivity = 10·50 m. mhos/cm.

TABLE 2

Analysis of the soil samples (original)

(Water-soluble cations and anions in m.e./litre of the solution, pH and electrical conductivity)

Treatments	Cations				Anions				
	Ca^{++}	Mg^{++}	Na^{++}	K^{+}	CO_3^{--}	Cl^{-}	SO_4^{--}	pH	Electrical conductivity m.mhos/cm. at 25° C
Soil alone	134·00	63·40	1506·34	0·66	250·40	490·14	520·10	9·00	10·50
Soil + Gypsum	155·50	60·14	1510·38	0·62	230·54	490·30	560·13	9·00	11·60
Soil + Superphosphate	134·00	63·40	1506·34	0·70	232·50	485·14	535·20	9·00	11·65
Soil + Cactus (Nag-phana H.)	165·38	62·28	1498·15	0·60	232·60	480·26	518·30	9·00	10·10
Soil + Cactus + Gypsum	170·45	59·84	1490·18	0·73	225·15	474·16	510·15	9·00	10·00
Soil + Cactus + Superphosphate	168·43	61·30	1495·16	0·72	229·18	478·18	514·35	9·00	10·05
Soil + Argemone–Mexicana	163·46	66·68	1497·10	0·61	235·30	483·50	518·90	9·00	10·15
Soil + Argemone–Mexicana + Gypsum	169·90	59·80	1492·30	0·73	227·19	480·34	511·40	9·00	10·06
Soil + Argemone–Mexicana + Superphosphate	168·50	61·45	1496·10	0·72	231·20	478·30	515·00	9·00	10·08

TABLE 3

Analysis of the soil samples taken after 50 *days of the treatment*

(Water-soluble cations and anions in me./litre of the solution, pH and electrical conductivity)

Treatments	Cations				Anions			pH	Electrical conductivity m. mhos/cm. at 25°C
	Ca^{++}	Mg^{++}	Na^{++}	K^{+}	CO_3^{--}	Cl^{-}	SO_4^{--}		
Soil alone	45·80	30·00	480·30	0·55	40·30	150·30	120·30	9·0	9·10
Soil + Gypsum	43·50	20·40	270·38	0·50	36·38	140·35	105·40	8·55	7·00
Soil + Superphosphate	44·30	21·00	300·10	0·53	37·00	142·40	110·30	8·70	7·10
Soil + Cactus (Nag-phana H.)	35·40	14·40	250·15	0·44	33·50	125·30	102·15	8·40	6·90
Soil + Cactus + Gypsum	27·15	12·34	210·34	0·44	28·30	113·15	97·10	8·00	6·80
Soil + Cactus + Superphosphate	29·50	13·60	225·10	0·45	29·40	118·20	98·50	8·15	6·85
Soil + Argemone–Mexicana	38·60	16·80	260·24	0·55	35·78	130·50	108·30	8·50	6·95
Soil + Argemone–Mexicana + Gypsum	30·10	14·50	220·30	0·55	30·65	115·10	100·45	8·05	6·87
Soil + Argemone–Mexicana + Superphosphate	32·20	15·90	234·20	0·56	31·18	122·15	102·66	8·10	6·92

TABLE 4

Analysis of soil samples taken after the harvest of the paddy crop

(Water soluble cations and anions in m.e/litre of the solution, pH and electrical conductivity)

Treatments	Cations				Anions			pH	Electrical conductivity m.mhos/cm. at 25°C
	Ca^{++}	Mg^{++}	Na^{++}	K^{+}	CO_3^{--}	Cl^{--}	SO_4^{--}		
Soil alone	20·15	14·10	360·38	0·34	22·10	95·20	90·50	8·50	8·50
Soil + Gypsum	16·20	11·15	200·40	0·30	18·25	85·30	75·30	8·05	5·80
Soil + Superphosphate	17·80	11·60	250·30	0·32	19·00	87·44	80·34	8·10	5·90
Soil + Cactus (Nag-phana H.)	12·30	8·50	180·38	0·22	15·20	70·40	71·05	8·00	4·80
Soil + Cactus + Gypsum	10·00	8·00	155·40	0·20	13·10	60·80	69·10	7·50	4·40
Soil + Cactus + Superphosphate	11·80	8·10	165·32	0·21	14·20	62·48	68·50	7·70	4·50
Soil + Argemone-Mexicana	13·20	9·40	190·50	0·34	16·80	75·36	75·45	8·10	5·00
Soil + Argemone-Mexicana + Gypsum	11·60	9·05	160·35	0·24	14·25	65·60	70·45	7·60	4·50
Soil + Argemone-Mexicana + Superphosphate	12·50	9·15	170·40	0·22	15·30	68·40	72·30	7·80	4·60

TABLE 5

Analysis of soil samples taken after the harvest of wheat crops.

(Water-soluble cations and anions in me/litre of the solution, pH and electrical conductivity

Treatments	Cations				Anions			pH	Electrical conductivity m. mhos/cm at 25°C
	Ca^{++}	Mg^{++}	Na^{++}	K^{+}	CO_3^{--}	Cl^{-}	SO_4^{--}		
Soil alone	15·00	10·30	150·40	0·30	15·20	40·30	35·00	8·45	6·40
Soil + Gypsum	10·30	6·70	90·52	0·25	13·30	34·38	30·14	8·00	3·10
Soil + Superphosphate	11·70	6·90	130·10	0·27	13·50	29·30	24·20	8·05	3·20
Soil + Cactus (Nag-phana H.)	7·80	4·68	85·15	0·22	10·40	20·15	15·30	7·95	2·50
Soil + Cactus + Gypsum	6·00	4·20	50·20	0·15	7·40	15·35	10·14	7·40	2·20
Soil + Cactus + Super-phosphate	6·50	4·30	60·30	0·17	7·80	17·20	20·30	7·55	2·30
Soil + Argemone–Mexicana	8·40	5·10	88·50	0·24	11·20	23·40	18·10	9·05	3·00
Soil + Argemone–Mexicana + Gypsum	6·70	4·50	55·30	0·18	8·00	17·30	12·08	7·50	2·90
Soil + Argemone–Mexicana + Superphosphate	7·00	4·65	65·30	0·20	8·20	18·60	13·10	7·60	2·40

TABLE 6

Yield of paddy grain and paddy straw

Treatments	Paddy grain in Kgm. Block						Paddy straw in Kgm. Block					
	A	B	C	D	Total	Yield/acre	A	B	C	D	Total	Yield/acre
Soil alone	0·95	1·10	1·15	1·12	4·4	88·00	2·10	2·50	2·20	2·40	9·20	184·00
Soil + Gypsum	2·90	2·50	2·30	2·60	10·00	200·00	4·00	4·10	3·90	4·20	16·20	324·00
Soil + Superphophate	1·80	1·85	1·70	1·95	7·30	146·00	3·20	3·00	2·80	2·60	11·60	222·00
Soil + Nag-phana H. (Cactus)	6·30	6·80	6·50	6·70	25·00	500·00	10·20	9·90	9·80	10·00	39·97	798·00
Soil + Cactus + Gypsum	10·20	9·80	10·10	9·90	40·00	800·00	15·10	14·80	15·00	14·90	59·80	1196·00
Soil + Cactus + Superphosphate	9·80	10·00	9·70	9·60	39·10	782·00	14·70	14·30	14·80	14·50	58·30	1166·00
Soil + Argemone–Mexicana	6·10	6·20	6·10	6·00	24·40	488·00	9·00	9·20	9·40	9·60	37·20	744·00
Soil + Argemone–Mexicana + Gypsum	9·00	8·90	8·80	8·60	35·30	706·00	12·40	12·30	12·50	12·00	49·20	984·00
Soil + Argemone–Mexicana + Superphosphate	8·50	8·60	8·20	8·70	34·00	680·00	11·20	11·00	11·10	11·30	44·60	892·00

TABLE 7

Treatments	Wheat grain in Kgm. Block						Wheat straw in Kgm. Block					
	A	B	C	D	Total	Yield/acre	A	B	C	D	Total	Yield/acre
Soil alone	1·40	1·50	1·70	1·50	6·10	122·00	4·10	4·00	3·90	4·20	12·20	144·00
Soil + Gypsum	2·70	2·80	2·50	2·00	10·60	212·00	5·00	5·30	5·10	4·90	20·30	406·00
Soil + Superphosphate	2·40	2·40	1·90	2·30	9·00	180·00	4·10	4·00	4·20	4·00	16·30	326·00
Soil + Nag-Phana H. (Cactus)	7·50	7·40	7·00	7·20	29·10	582·00	14·80	14·60	14·50	14·50	60·40	1208·00
Soil+Cactus +Gypsum	10·50	10·60	10·80	8·70	42·60	852·00	20·00	20·20	20·10	20·30	80·60	1612·00
Soil + Cactus + Superphosphate	10·20	10·40	10·20	10·30	41·10	822·00	19·80	19·60	19·00	19·10	77·50	1550·00
Soil + Argemone–Mexicana	7·00	7·10	7·00	7·20	28·30	566·00	14·00	13·80	14·10	13·70	55·30	1112·00
Soil + Argemone–Mexicana + Gypsum	9·50	9·80	9·60	9·80	38·70	774·00	17·20	17·60	17·30	17·50	69·60	1392·00
Soil + Argemone–Mexicana+Superphosphate	9·50	9·10	9·10	9·50	37·20	744·00	17·10	17·20	16·90	16·80	68·00	1360·00

TABLE 8

Yield of paddy grain and paddy straw (second crop)

Treatments	Paddy grain in Kgm. Block						Paddy straw in Kgm. Block					
	A	B	C	D	Total	Yield/acre	A	B	C	D	Total	Yield/acre
Soil alone	1·60	1·50	1·80	1·60	6·50	130·00	2·50	2·40	2·80	2·50	10·20	204·00
Soil + Gypsum	2·50	2·70	2·60	2·40	10·00	200·00	4·00	3·80	3·70	3·90	15·40	308·00
Soil + Superphosphate	2·30	2·30	1·80	2·20	8·60	172·00	3·10	3·00	3·10	2·80	12·00	240·00
Soil + Nag-Phana H. (Cactus)	12·10	12·00	11·90	12·10	48·10	962·00	17·50	17·60	17·30	17·20	69·60	1392·00
Soil + Cactus + Gypsum	15·00	15·20	15·10	15·00	60·30	1206·00	22·00	22·10	21·90	22·00	88·00	1760·00
Soil + Cactus + Superphosphate	14·70	14·30	14·00	14·10	57·10	1142·00	21·10	21·20	21·00	21·10	84·40	1688·00
Soil + Argemone-Mexicana	11·30	11·70	11·40	11·80	46·20	924·00	16·20	16·40	16·80	16·80	66·20	1324·00
Soil + Argemone-Mexicana + Gypsum	14·90	14·40	14·50	14·50	58·30	1166·00	21·50	21·60	21·80	21·70	86·60	1732·00
Soil + Argemone-Mexicana + Superphospate	11·00	11·10	10·80	10·80	43·80	976·00	20·10	20·00	20·00	20·10	80·20	1604·00

It is concluded that as Nag-phana H. (Cactus) can be grown in the field with little attention and care, especially with very low water requirements, this can be very easily grown and incorporated into an alkali land along with gypsum for a good growth and development of paddy and wheat and even a second crop of either of the two with increased yield and without any further addition either of Nag-phana H. (Cactus) or gypsum.

References

De, S. K. *Practical Agricultural Chemistry*, 1965.

Dhar, N. R. *Proc. nat. Acad. Sci. India*, **25** A, 211-327, 1955.

Dalton, J. D., Russell, G. C. and Sieling, Dale H. *Soil Sci.*, **73**, 173–181, 1952.

Leather, J. W. *Govt India rep., Allahabad*, **88**, 1914.

Makarets, I K. *Izv. Fimir XazerS-kh. Akad.*, **5**, 43–62, 1957.

Overstreet, R., Marton, J. C. and King, H. M. *Hilgardia*, **21**, 113–137, 1951.

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Electro-magnetic field in an isotropic space-time

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In a four dimensional space-time with signature (+ − − −), we consider an isotropic line-element and working directly with the skew-symmetric electromagnetic field tensor, we find the most general solution of the Einstein-Maxwell field equations for a real, source-free, electromagnetic field in the presence of a gravitational field.

1. The gravitational behaviour of classical electromagnetic field is governed by the Einstein-Maxwell equations[1],

$$G_{\mu}^{\nu} = 8\pi\left(F_{\mu\sigma}F^{\sigma\nu} + \tfrac{1}{4}\,g_{\mu}^{\nu}\,F_{\sigma\varepsilon}F^{\sigma\varepsilon}\right) \tag{1·1}$$

$$F_{\mu\nu,\sigma} + F_{\nu\sigma,\mu} + F_{\sigma\mu,\nu} = 0 \tag{1·2}$$

$$(\sqrt{-g}\;F^{\mu\nu})_{,\nu} = 0 \tag{1·3}$$

where G_{μ}^{ν} is the Einstein tensor,

$$G_{\mu}^{\nu} = R_{\mu}^{\nu} - \tfrac{1}{2}\,g_{\mu}^{\nu}\,R \tag{1·4}$$

R_{μ}^{ν} being the Ricci curvature tensor,

$$R_{\mu\nu} = \Gamma^{\sigma}_{\mu\tau}\,\Gamma^{\tau}_{\sigma\nu} - \Gamma^{\sigma}_{\mu\nu}\,\Gamma^{\tau}_{\sigma\tau} + \left(\Gamma^{\tau}_{\mu\tau}\right)_{,\nu} - \left(\Gamma^{\tau}_{\mu\nu}\right)_{,\tau} \tag{1·5}$$

and

$$R \underset{=}{\text{def}} R_{\mu\nu}\,g^{\mu\nu} \tag{1·6}$$

A comma indicates partial differentiation with respect to the index following it, $g^{\mu\nu}$ is the fundamental tensor, $\Gamma^{\sigma}_{\mu\nu}$ are the Christoffel 3-index symbols, and $F_{\mu\nu}$ is the skew-symmetric tensor whose components give the electric intensity magnetic induction associated with the field.

A non-null electro-magnetic field implies that the square of the Maxwell tensor is positive. That is,

$$\rho^2 \equiv \tfrac{1}{4}\,R_{\sigma}^{\rho}\,R_{\rho}^{\sigma} > 0 \tag{1·7}$$

When the equations (1·1) to (1·3) are satisfied, the Rainich algebraic conditions for a real, source-free, electro magnetic field in the presence of a gravitational field are as follows[2].

The Maxwell tensor has zero trace,

$$R = 0 \tag{1·8}$$

The square of this tensor is proportional to the unit matrix,

$$R^{\sigma}_{\mu} R^{\nu}_{\sigma} = \rho^2 \delta^{\nu}_{\mu} \tag{1·9}$$

The electro-magnetic energy-density is positive-definite,

$$R_{00} > 0 \tag{1·10}$$

A certain form of the Ricci tensor has zero curl.

$$\frac{g_{\mu\sigma}\, \varepsilon^{\sigma\rho\nu\lambda}\, R_{\rho\psi}\, R\psi_{\nu;\lambda}}{4\rho^2 \sqrt{-g}} \equiv a_{,\mu} = 0 \tag{1·11}$$

where $\varepsilon^{\sigma\rho\nu\lambda}$ is the Levi-Civita permutation symbol with $\varepsilon^{0123} = 1$

Further, for an electromagnetic field, the eigen values of the Einstein tensor reduce to the form,

$$\rho, \rho, -\rho, -\rho$$

2. We consider the isotropic line element[3].

$$ds^2 = e^{-2\phi}\, dt^2 - e^{2\phi}\, (dx^2 + dy^2 + dz^2) \tag{2·1}$$

where we take ϕ to be a function of x, y and z.

We have,

$$g_{11} = g_{22} = g_{33} = -e^{2\phi}, \quad g_{00} = e^{-2\phi}$$

$$g^{11} = g^{22} = g^{33} = -e^{-2\phi}, \quad g^{00} = e^{-2\phi} \tag{2·2}$$

and $$g^{\mu\nu} = g_{\mu\nu} = 0 \text{ for } \mu \neq \nu$$

We find the exact solutions of the equations (1·1) to (1·3) by working directly with the components of $F_{\mu\nu}$, without making use of a 4-potential from which $F_{\mu\nu}$ can be derived.

For the line element (2·1) the non vanishing $\Gamma^{\lambda}_{\mu\nu}$ are found to be

$$\begin{aligned}
&\Gamma^{1}_{11} = \Gamma^{2}_{12} = \Gamma^{3}_{13} = -\Gamma^{1}_{33} = -\Gamma^{0}_{01} = -\Gamma^{1}_{22} = a \\
&\Gamma^{1}_{12} = \Gamma^{2}_{22} = \Gamma^{3}_{23} = -\Gamma^{2}_{11} = -\Gamma^{2}_{33} = -\Gamma^{0}_{02} = \beta \\
&\Gamma^{1}_{13} = \Gamma^{3}_{33} = \Gamma^{2}_{23} = -\Gamma^{2}_{11} = -\Gamma^{2}_{22} = -\Gamma^{0}_{03} = \gamma \\
&\Gamma^{1}_{00} \,/\, \alpha = \Gamma^{2}_{00} \,/\, \beta = \Gamma^{3}_{00} \,/\, r = -e^{-4\phi}.
\end{aligned} \tag{2·3}$$

where $\alpha \equiv \frac{\partial \phi}{\partial x}$, $\beta \equiv \frac{\partial \phi}{\partial y}$ and $\gamma \equiv \frac{\partial \phi}{\partial z}$

Since $R = 0$

$$G_{\mu}^{\nu} = R_{\mu}^{\nu}$$

and from (1·3), we get,

$$(2\cdot4) \qquad G_{\mu}^{\nu} = e^{-2\phi} \begin{bmatrix} -\xi^2 - 2\nabla^2\phi & O & O & O \\ O & 2\alpha^2 - \xi^2 & 2\alpha\beta & 2\alpha\gamma \\ O & 2\alpha\beta & 2\beta^2 - \xi^2 & 2\beta\gamma \\ O & 2\alpha\gamma & 2\beta\gamma & 2\gamma^2 - \xi^2 \end{bmatrix}$$

where $\xi^2 = \alpha^2 + \beta^2 + \gamma^2$.

We observe that in the case of empty space, ϕ is constant and since this constant can be taken to be zero without loss of generality, (2·1) reduces to the metric of flat space-time.

Now, from (1·6) and (1·8),

$$(2\cdot5) \qquad \nabla^2\phi + \xi^2 = 0$$

which is the Laplacian equation in Cartesian coordinates.

Therefore, from (2·4),

$$(2\cdot6) \qquad R_{00} = \xi^2 > 0$$

The eigen values of the Einstein tensor (2·4) are $(\rho, \rho, -\rho, -\rho)$ where,

$$(2\cdot7) \qquad \rho = \xi^2 e^{-2\phi}$$

3. From (1·1), (2·2) and (2·4) we get the following relations.

$$(3\cdot1) \qquad g^{22}\left(F_{12}^2 + F_{13}^2 + F_{23}^2\right) + g^{00}\left(F_{01}^2 + F_{02}^2 + F_{03}^2\right) = \frac{1}{4\pi}(\alpha^2+\beta^2+\gamma^2)$$

$$(3\cdot2) \qquad g^{22}\left(F_{12}^2 + F_{13}^2 - F_{23}^2\right) - g^{00}\left(F_{02}^2 + F_{03}^2 - F_{01}^2\right) = \frac{1}{4\pi}(\alpha^2 - \beta^2 - \gamma^2)$$

$$(3\cdot3) \qquad g^{22}\left(F_{12}^2 + F_{23}^2 - F_{13}^2\right) - g^{00}\left(F_{01}^2 + F_{03}^2 - F_{02}^2\right) = \frac{1}{4\pi}(\beta^2 - \alpha^2 - \gamma^2)$$

$$(3\cdot4) \qquad g^{22}\left(F_{13}^2 + F_{23}^2 - F_{12}^2\right) - g^{00}\left(F_{01}^2 + F_{02}^2 - F_{03}^2\right) = \frac{1}{4\pi}(\gamma^2 - \alpha^2 - \beta^2)$$

$$(3\cdot5) \qquad g^{33} F_{13} F_{23} + g^{00} F_{01} F_{02} = \frac{1}{4\pi}\alpha\beta$$

$$(3\cdot6) \qquad g^{22} F_{12} F_{13} - g^{00} F_{01} F_{03} = -\frac{1}{4\pi}\alpha\gamma$$

$$(3\cdot7) \qquad g^{11} F_{12} F_{13} + g^{00} F_{02} F_{03} = \frac{1}{4\pi}\beta\gamma$$

(3·8) $$F_{13}\,F_{03}+F_{12}\,F_{02}=0$$

(3·9) $$F_{12}\,F_{01}-F_{23}\,F_{03}=0$$

(3·10) $$F_{13}\,F_{01}+F_{23}\,F_{02}=0$$

4. From (3·2) and (3·3), adding,

(4·1) $$g^{00}F_{03}^{2}-g^{22}F_{12}^{2}=\frac{1}{4\pi}\gamma^{2}$$

From (3·3) and (3·4),

(4·2) $$g^{00}F_{01}^{2}-g^{22}F_{23}=\frac{1}{4\pi}\alpha^{2}$$

From (3·2) and (3·4),

(4·3) $$g^{00}F_{02}^{2}-g^{22}F_{13}^{2}=\frac{1}{4\pi}\beta^{2}$$

Equations (4·2) and (3·9) give,

(4·4) $$\frac{F_{01}}{\alpha}=\frac{F_{03}}{\gamma}$$

Similarly equations (3·8) and (4·1) give,

(4·5) $$\frac{F_{02}}{\beta}=\frac{F_{03}^{2}}{\gamma}$$

From (4·4) and (4·5) we now get,

(4·6) $$\frac{F_{01}}{\alpha}=\frac{F_{02}}{\beta}=\frac{F_{03}}{\gamma}=a\,p,\ \text{say},$$

where a is a constant and p is some function of (x, y, z) to be determined.

Similarly from (3·5), (3·10) and (3·8), we get,

(4·7) $$\frac{F_{12}}{\gamma}=\frac{F_{23}}{\alpha}=-\frac{F_{13}}{\beta}=b\,q,$$

where b is a constant and q another function of (x, y, z) to be determined.

Substituting from (4·6) and (4·7) into (3·5) and taking $a=-b=\frac{1}{\sqrt{4\pi}}$ we get,

(4·8) $$g^{00}p^{2}-g^{33}q^{2}=1$$

From (1·2) we have,

(4·9) $$q\left(\frac{\partial\alpha}{\partial x}+\frac{\partial\beta}{\partial y}+\frac{\partial\gamma}{\partial z}\right)+\alpha\frac{\partial q}{\partial x}+\beta\frac{\partial q}{\partial y}+\gamma\frac{\partial q}{\partial z}=0$$

(4·10) $$\alpha\frac{\partial p}{\partial y}-\beta\frac{\partial p}{\partial x}=0$$

(4·11) $$\gamma \frac{\partial p}{\partial x} - \alpha \frac{\partial p}{\partial z} = 0$$

(4·12) $$\beta \frac{\partial p}{\partial z} - \gamma \frac{\partial p}{\partial y} = 0$$

Differentiating (4·8) partially with respect to x and y and using (4·9), we get,

$$\alpha \frac{\partial q}{\partial y} = \beta \frac{\partial q}{\partial x}$$

(4·13) $$\beta \frac{\partial q}{\partial z} = \gamma \frac{\partial q}{\partial y}$$

$$\gamma \frac{\partial q}{\partial x} = \alpha \frac{\partial q}{\partial z}$$

From (4·10), (4·11), (4·12), and (4·13), solving for q,

(4·14) $$q = ce^{\phi}$$

where c is a constant.

(4·8) now gives,

(4·15) $$p = \sqrt{1-c^2}\, e^{-\phi}$$

Thus the components of $F_{\mu\nu}$ are found to be,

(4·16) $$F_{\mu\nu} = \frac{1}{\sqrt{4\pi}} \begin{bmatrix} 0 & -\alpha\sqrt{1-c^2}\, e^{-\phi} & -\beta\sqrt{1-c^2}\, e^{-\phi} & -\gamma\sqrt{1-c^2}\, e^{-\phi} \\ \alpha\sqrt{1-c^2}\, e^{-\phi} & 0 & r\,c\,e^{\phi} & \beta\,c\,e^{\phi} \\ \beta\sqrt{1-c^2}\, e^{-\phi} & -\gamma\,c\,e^{\phi} & 0 & \alpha\,c\,e^{\phi} \\ \gamma\sqrt{1-c^2}\, e^{-\phi} & -\beta\,c\,e^{\phi} & -\alpha\,c\,e^{\phi} & 0 \end{bmatrix}$$

It can be easily verified that for these $F_{\mu\nu}$ the Einstein-Maxwell equations (1·3) are also satisfied.

Thus we obtain the most general form of the solutions of Einstein-Maxwell equations in the static case, for an isotropic space-time.

References

1. Introduction to general Relativity *by* Adler. *Bazin and Schiffer International series*, 396, 1965.
2. Recent developments in General Relativity. *Polish Scientific Publishers, Pergamon Press*, 384, 1962.
3. Evidence for Gravitational Theories. *Proc. Enrico Fermi School of Physics, Italy, Academic Press*, 237, 1961.

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Radial Oscillations of a Particular Magnetic stellar Model

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Abstract

In this paper we have considered the radial oscillations of a particular stellar model having a point mass equivalent to one third of the total mass of the star at the centre and homogeneous throughout the rest.

The theory of adiabatic radial oscillations was developed by Eddington[1], who applied it to find the period of oscillations of polytropic models. This theory was applied in the cases of three different models with specified density distributions by Sterne[2] and he was able to obtain algebraic expressions for the amplitude of oscillations. The presence of magnetic field of 10^{-6} gauss in the galactic plane follows by the study of the properties of cosmic rays and observed polarization of light from distant stars. The magnetic fields of the order of one gauss are observed to exist in the earth's and Sun's surface and magnetic fields of the order of a thousand gauss have been observed in the reversing layers of some stars. Thus we come to the conclusion that the magnetic field play an increasing role in the study of the galactic system We choose a simple form of magnetic field, as all the stars have magnetic field probably in the interior or in the atmosphere.

This steller model was first considered by Chandrika Prasad[3] and he established the stability of the radial oscillations of this model taking the magnetic field to be absent. We have considered that the radial oscillations will continue to remain stable when the model is taken to be a magnetic stellar model.

In the case of spherical Magnetic stars the differential equations for small radial oscillations, when a simple form of magnetic field is chosen, is given by

$$\frac{d^2a_1}{dx^2}+\left\{\frac{4}{x}-\frac{g_0\rho_0R}{P_0}+\frac{(\gamma+1)(q+2)\mu l^2}{4\pi\gamma P_0R^{q+4}.\ x^{2q+5}}\right\}\frac{da_1}{dx}$$
$$+\left[\frac{n^2\rho_0R^2}{\gamma P_0}-\frac{(3\gamma-4)}{\gamma P_0}\frac{g_0\rho_0R}{x}-\frac{(q+2)(2q+4-3\gamma)\mu l^2}{4\pi\gamma P_0R^{2q+4}.\ x^{2q+6}}\right]a_1=0 \qquad (1)$$

where n is the frequency of oscillation, R is the radius of the star in the undisturbed state and x lies between zero and one such that $\xi_0 = Rx$, γ is the effective ratio of specific heats (regarding the matter and enclosed radiation as one system) P_0, g_0, ρ_0 and H_0 are the undisturbed values of pressure, gravity, density and the strength of magnetic field. μ is permeability and a hydrostatic pressure due to magnetic field is $\mu H^2/8\pi$, where H is given by the equation $H = 1/\xi^{q+2}$. Equation (1) is the very equation which we also obtained in the case of the radial oscillations of a slowly rotating-magnetic star[4] in which case the variables ρ_0, g_0, P_0 however denote the undisturbed values for the spheroid. And in that paper we came to the conclusion that the superposition of a small amount of rotation does not materially

alter the results obtained for small oscillations of the spherical stars in the first order. The boundary conditions here will also be the same as in the case of spherical magnetic stellar model[5]. For the model in question we have

$$\left.\begin{aligned} M(\xi_0) &= \frac{2}{3}\pi\rho_0 R^3 + \frac{4}{3}\pi\xi_0^3\rho_0 \\ g_0 &= \frac{2}{3}\pi G\rho_0 \frac{R^3}{\xi_0^2} + \frac{4}{3}\pi G\rho_0\xi_0 \\ &= \frac{2}{3}\pi G\rho_0 R\left(\frac{1}{x^2} + 2x\right) \end{aligned}\right\} \quad (2)$$

and $$P_0 = \frac{2}{3}\pi G\rho_0^2 R^2\left(\frac{1-x^3}{x}\right) - \frac{\mu l^2}{8\pi R^{2q+4}}\left(\frac{1-x^{2q+4}}{x^{2q+4}}\right)$$

Substituting these values in the equation (1) we obtain

$$\begin{aligned} &\left\{\frac{2}{3}\pi G\rho_0^2 R^2\left(\frac{1-x^3}{x}\right) - \frac{\mu l^2}{8\pi R^{2q+4}}\left(\frac{1-x^{2q+4}}{x^{2q+4}}\right)\right\}\frac{d^2a_1}{dx^2} \\ &+\left[\frac{4}{x}\left\{\frac{2}{3}G\rho_0^2 R^2\left(\frac{1-x^3}{x}\right) - \frac{\mu l^2}{8\pi R^{2q+4}}\left(\frac{1-x^{2q+4}}{x^{2q+4}}\right)\right\}\right. \\ &\left. - \frac{2}{3}G\pi\rho_0 R\left(\frac{1+2x^3}{x^2}\right)R\rho_0 - \frac{(\gamma+1)(q+2)\mu l^2}{4\pi\gamma R^{2q+4}\cdot x^{2q+5}}\right]\frac{da_1}{dx} \\ &+\left[\frac{n^2\rho_0 R^2}{\gamma} - \frac{(3\gamma-4)}{\gamma x}\rho_0 R\cdot\frac{2}{3}G\pi\rho_0 R\frac{(1+2x^3)}{x^2}\right. \\ &\left. - \frac{(q+2)(2q+4-3\gamma)\mu l^2}{4\pi\gamma R^{2q+4}\cdot x^{2q+6}}\right]a_1 = 0 \end{aligned} \quad (3)$$

Now denoting $\triangle_1, \triangle_2, \triangle_3,$ as follows

$$\left.\begin{aligned} &\triangle_1 = 16\pi^2 G\gamma\rho_0^2 R^{2q+6},\ \triangle_2 = 3\gamma\mu l^2 \\ &\triangle_3 = 16\pi^2\gamma G\rho_0^2 R^{2+q+6},\ \triangle_4 = 3\gamma\mu l^2 = \triangle_2 \\ &\triangle_5 = 64\gamma\pi^2 G\rho_0^2 R^{2q+6},\ 4\triangle_1,\ \triangle_6 = 3\mu\gamma l^2 = \triangle_2 \\ &\triangle_7 = 4\triangle_3 = 4\triangle_1,\ \triangle_8 = 6(\gamma q + q + 2)\mu l^2 - 3\mu l^2\gamma \\ &\triangle_9 = 8\pi\rho_0 R^{2q+6}\{4(3\gamma-4)\rho_0 G - 3n^2\} \\ &\triangle_{10} = 16(3\gamma-4)G\pi^2\rho_0^2 R^{2q+6}\} \\ &\triangle_{11} = 6(q+2)(2q+4-3\gamma)\mu l^2. \end{aligned}\right\} \quad (4)$$

equation (3) reduces to

$$\begin{aligned} &x^2\{\triangle_1 x^{2q+6} - \triangle_2 x^{2q+4} - \triangle_3 x^{2q+3} + \triangle_4\}\, d^2a_1/dx^2 \\ &+ x\{\triangle_5 x^{2q+6} - \triangle_6 x^{2q+4} - \triangle_7 x^{2q+3} - \triangle_8\}\, da_1/dx \\ &+ \{\triangle_9 x^{2q+6} + \triangle_{10} x^{2q+3} + \triangle_{11}\}\, a_1 = 0 \end{aligned} \quad (5)$$

When we put $x = 1$ in the above equation we obtain the equation at the boundary.

(5) is a differential equation of second order and its indicial equation has two distinct roots $2(q+2)$ and $\{2(n+2)-3\}/\gamma$. The differential equation has got regular singularities at $x = 0$ and $x = 1$.

We assume the following series solution for (5)

$$a_1 = \sum_{0}^{\infty} b_\lambda x^{c+\lambda} \tag{6}$$

substituting the series in (5) and equating the coefficients of the lowest power of x, we get the indicial equation whose roots are given above.

Hence expanding the solution about the origin in a power series of the form (6) and assuming $b_0 = 1$ (as the equation is linear) we get the recurrence formula as

$$\begin{aligned} & b_\lambda \{(c+\lambda)(c+\lambda-1)\Delta_1 + (c+\lambda)\Delta_5 + \Delta_9\} \\ & - b_{\lambda+2}\{(c+\lambda+2)(c+\lambda+1)\Delta_2 + (c+\lambda+2)\Delta_6\} \\ & - b_{\lambda+3}\{(c+\lambda+3)(c+\lambda+2)\Delta_3 + (c+\lambda+3)\Delta_7 - \Delta_{10}\} \\ & + b_{\lambda+2q+6}\{-(c+\lambda+2q+6)\Delta_8 + (c+\lambda+2q+6) \\ & \qquad (c+\lambda+2q+5)\Delta_4 - \Delta_{11}\} = 0 \end{aligned} \tag{7}$$

Arranging in powers of λ and dividing by b_λ we get

$$\begin{aligned} & \lambda^2\left[\Delta_1 - \frac{b_{\lambda+2}}{b_\lambda}\Delta_2 - \frac{b_{\lambda+3}}{b_\lambda}\Delta_3 + \frac{b_{\lambda+2q+6}}{b_\lambda}\Delta_4\right] \\ & + \lambda\left[(2c-1)\Delta_1 + \Delta_5 - \{(2c+3)\Delta_2 + \Delta_6\}\frac{b_{\lambda+2}}{b_\lambda}\right. \\ & \left. - \{2c+5)\Delta_3 + \Delta_7\}\frac{b_{\lambda+3}}{b_\lambda} - \{(2c+4q+11)\Delta_4 - \Delta_8\}\frac{b_{\lambda+2q+6}}{b_\lambda}\right] \\ & + \left[(c^2-c)\Delta_1 + c\Delta_5 + \Delta_9 - \frac{b_{\lambda+2}}{b_\lambda}\{(c^2+3c+2)\Delta_2 + (c+2)\Delta_6\}\right. \\ & \qquad - \frac{b_{\lambda+3}}{b_\lambda}\{(c^2+5c+6)\Delta_3 + (c+3)\Delta_7 - \Delta_{10}\} \\ & + \frac{b_{\lambda+2q+6}}{b_\lambda}\{c^2 + \overline{4q+11}\,c + 4q^2 + 22q + 30)\Delta_4 \\ & \left. \qquad - (c+2q+6)\Delta_8 + \Delta_{11}\}\right] = 0 \end{aligned} \tag{8}$$

Dividing by λ^2 and proceeding to the limit as $\lambda \to \infty$ we have

$$\Delta_1 - \Delta_2 \lim_{\lambda\to\infty} \frac{b_{\lambda+2}}{b_\lambda} - \Delta_3 \lim_{\lambda\to\infty} \frac{b_{\lambda+3}}{b_\lambda} + \Delta_4 \lim_{\lambda\to\infty} \frac{b_{\lambda+2q+6}}{b_\lambda} = 0 \tag{9}$$

Let $\lim_{\lambda\to\infty} \frac{b_{\lambda+1}}{b_\lambda} = k$ (10)

Hence (9) becomes

$$\Delta_1 - \Delta_2 k^2 - \Delta_3 k^3 + \Delta_4 k^{2q+6} = 0 \tag{11}$$

Equation (11) is satisfied by $k = 1$. Hence (10) may be written in the form

$$\lim_{\lambda\to\infty} \frac{b_{\lambda+1}}{b_\lambda} = k = 1 \tag{12}$$

Thus the series solution for (5) has unit radius of convergence. The series solution (6) is convergent in the neighbourhood of the origin because (5) has regular singularities at $x = 0$ and $x = 1$. Hence the series solution will retain its convergence upto the next singularity $x = 1$.

We proceed to test the convergence of (7) at $x = 1$.

$$\text{Now let } \frac{b_{\lambda+1}}{b_\lambda} = 1 - \varepsilon \tag{13}$$

Where ε is a function of λ such that

$$\varepsilon = 0\left(\frac{1}{\lambda^{p'}}\right) \tag{14}$$

p' being positive integer.

Then we have

$$\left.\begin{aligned} \frac{b_{\lambda+2q+6}}{b_\lambda} &= (1-\varepsilon)^{2q+6} = 1 - (2q+6)\,\varepsilon \\ \frac{b_{\lambda+3}}{b_\lambda} &= (1-\varepsilon)^3 = 1 - 3\varepsilon \\ \frac{b_{\lambda+2}}{b_\lambda} &= (1-\varepsilon)^2 = 1 - 2\varepsilon \\ &\text{to the first power of } \varepsilon. \end{aligned}\right\} \tag{15}$$

From (15) and (8) we have, keeping terms of the highest order

$$\lambda^2[\,\Delta_1 - (1-2\varepsilon)\,\Delta_2 - (1-2\varepsilon)\,\Delta_3 + \{1-(2q+6)\varepsilon\}\,\Delta_4]$$
$$+ \lambda[\,(2c-1)\,\Delta_1 + \Delta_5 - \{(2c+3)\,\Delta_2 + \Delta_6\} - \{(2c+5)\Delta_3 + \Delta_7\}$$
$$+ \{(2c+4q+11)\,\Delta_4 - \Delta_8\}\,] = 0 \tag{16}$$

We know that $\Delta_1 - \Delta_2 - \Delta_3 + \Delta_4 = 0$ (17)

Hence (16) reduces to

$$2\lambda^2[\,\Delta_2 + \Delta_3 - (q+3)\Delta_4]\,\varepsilon + \lambda[-\Delta_1 + \Delta_5 - 3\Delta_2 - \Delta_6 - 5\Delta_3 - \Delta_7$$
$$+ (4q+11)\Delta_4 - \Delta_8] = 0 \tag{18}$$

which obviously gives ε to be of the form

$$\varepsilon = \frac{Q}{\lambda} \tag{19}$$

Hence we have

$$\frac{b_{\lambda+1}}{b_\lambda} = 1 - \frac{Q}{\lambda} + O\left(\frac{1}{\lambda^2}\right) \tag{20}$$

which shows that the series solution (6) is also convergent for $x = 1$. (Gauss's Rules).

Thus we conclude that a magnetic star having a point mass equivalent to one third of the whole mass of the star and the centre and homogeneous throughout the rest, if sets oscillating the small radial oscillations will remain stable *i.e.* the star will continue to oscillate.

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References

1. Eddington, A. S. The Internal Constitution of the Stars.
2. Sterne, T. E. *M. N. R. A. S.* **97**, 583, 1937.
3. Prasad, C. *M. N.*, **108**, 414, 1948.
4. Gupta, R. S. and Upadhyaya, U. S. *Progress of Mathematics*, 2(1), 167–180, 1968.
5. Gurtu, V. K. and Banerji, A. C. *Proc. Nat. Aca. Sc. India*, **36** A pp. 129–140.
6. Bromwich, T. J. "An Introduction to the theory of Infinite series."

Proc. Nat. Acad. Sci., India, 39(A), III, 1969.

Determination of Cadmium with Isoquinoline

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Many heterocyclic Nitrogen compounds are being used as analytical reagents in the inorganic analysis *e g.* pyridine gives well defined insoluble complexes with Copper, Cadmium, Zinc, Cobalt and Nickel in the presence of thiocyanate thus determined gravimetrically[1]. The reaction of pyridine with aqueous solution of the metals in the presence of thiocyanate is not specific. Only individua member can be determined. Isoquinoline also gives similar complexes in the prlesence of thiocyanate with the above metals but at various acidities. Hence it is specific for individual metals at a particular pH and its analytical properties were studied by A. Dolphe Spakowski and Henry Freiser[2], they used it for the determination of Copper and Zinc gravimetrically. In the present communication, this method has been extended for the gravimetric determination of Cadmium and a volumetric method is also given in which a known excess of thiocyanate is being added to the aqueous solution of the metal nitrate in the presence of isoquinoline solution at a proper pH and the excess is determined with standard solution of Mercuric Nitrate using diphenyl carbazone as indicator.

As the isoquinoline is insoluble in water its solution in formic acid is prepared (since the mineral acids interfere with the indicator). This solution of isoquinoline in formic acid precipitates copper at pH = 2·6, Zinc = 3·7, Cadmium = 3·4, Nickel and Cobalt give turbidity at pH = 4·0. Thus the separation of Copper from Cadmium, Zinc, Nickel and Cobalt can be possible but the separation of Zinc from Cadmium can not be possible as Zinc gives turbid ity at = 3·25.

Experimental

Reagents.—0·2 N Potassium Thiocyanate was prepared from B.D.H. analare sample and standarised by Volhard's method[3]. 0·1 N solution of Mercuric Nitrat- was prepared from E Mercks sample and standardised with Potassium Thiocy anate using Diphenyl Carbazone as indicator. 0·05 M solution of Cadmium Nitrate prepared from Analar sample and the metal contents were controlled by classical methods. 1 M solution of iso-quinoline was prepared by dissolving 12·9 gms of pure sample of isoquinoline in 4–5 mls of 85% formic acid and the volume made to 100 mls with distilled water. A saturated solution of the indicator Diphenyl Carbazone was prepared in alcohol.

Equipment.—Marconi pH meter was used for measuring the various pH.

Procedure.—5 mls of ·05 M Cadmium Nitrate solution was taken in 100 ml dry beaker, 10 mls of 0·2 N Potassium thiocyanate so'ution and 2 mls of isoquinoline solution. The pH was adjusted to = 4 with dilute solution of Sodium Hydroxide added drop-wise and the volume was made to 25 mls exact. Warmed in water bath for 2–3 minutes and allowed to stand for 10–20 minutes, filtered through

Whatman No. 42 filter pap r. The first few mls were rejected, 5 mls of the filtrate was taken in the titration flask added 20 mls of distilled water. 1 ml of dilute solution of formic acid (10 mls of 85% formic acid was diluted to 100 mls) in order to bring the pH of the filterate = 3·1 where the end point was quite sharp. 1–2 drops of the indicator was added and titrated against 0·1 N mercuric nitrate solution. The end point was marked with the appearance of violet colour.

The effect of pH on the precipitation of Cadmium is given in Table No. I.

TABLE 1

Volume of Cadmium soln. taken in mls.	Amount of Cadmium taken in mg.	pH of the solution	Amount of Cadmium found in mg.
4·00	22·50	5·70	22·490
4·00	22·50	5·00	22·490
4·00	22·50	4·50	22·490
4·00	22·50	3·70	22·490
4·00	22·50	3·40	22·490
4·00	22·50	3·30	18·240
4·00	22·50	3·00	6·500
4·00	22·50	2·90	–

The pH of the solution can be increased with dilute solution of Sodium Hydroxide where it can be decreased with dilute solution of formic acid.

Typical set of results are shown in Table No. 2 at pH = 4·4.

TABLE 2

Strength of Potassium thiocynate = ·2 N

Strength of Mercuric nitrate = ·1 N

10 mls of Potassium thiocyanate was used in each determination

Volume of Cadmium soln. taken in mls.	Amount of Cadmium taken in mg.	Volume of iso-quinoline added in mls.	Volume of mercuric nitrate used for 5 mls of residual thiocyanate in mls	Volume of thiocyanate consumed in mls.	Amount of Cadmium found in mg.
1·00	5·625	·5	3·80	·50	5·620
2·00	11·250	1·00	3·60	1·00	11·245
2·60	14·625	1·00	3·48	1·30	14·620
4·50	25·3125	2·00	3·10	2·250	20·290
6·00	33·750	2·00	2·80	3 00	33·730

In the gravimetric method the reagents were the same as used in the volumetric method. In the case of large amount of Cadmium (100–200 mg) the complex Cd $(C_9H_7N)_2$ $(SCN)_2$ was allowed to stand for two hours, filtered over a sintered glass G–4 filteration crucible, washed with distilled water, dried at 90–100°C for about one hour and weighed as Cd $(C_9H_7N)_2$ $(SCN)_2$. The results are given in Table No. 3 at pH. 4·50.

TABLE 3

Volume of Cadmium soln. taken in mls.	Amount of Cadmium taken in gms.	Volume of iso-quinoline added in mls.	Weight of precipitate in gms.	Cadmium found in gms.
20·00	·1125	6·00	·4853	·1123
24·00	·1350	6·00	·5920	·1346
28·00	·1575	7·00	·6800	·1573
32·00	·1800	8·00	·7768	·1797
40·00	·2250	10·00	·9742	·2254

Interference.—In the volumetric method Pb^{+2}, Sn^{+2}, Sn^{+4}, Al^{+3}, Bi^{+3}, Cr^{+3} and Mn^{+2} do not interfere where Ag^{+}, Hg^{+2}, Co^{+2}, Ni^{+2}, Cu^{+2}, Zn^{+2} and Fe^{+3} interfere. Where in the gravimetric method Ag^{+}, Hg^{+2}, Co^{+2}, Ni^{+2}, Zn^{+2}, Cu^{+2} and Bi^{+3} interfere. On the whole the method is quite simple, less time-consuming as compared to other volumetric and gravimetric methods and is equally accurate. The conditions and the reagents are also simple.

Summary

The method is based upon the fact that the metal such as Cu, Co, Ni, Zn and Cd form well defined insoluble complexes quantitatively at different pH when their aqueous solutions are treated with a known excess of thiocyanate in the presence of iso-quinoline solution and the excess of thiocyanate is determined mercurimetrically using diphenyl carbazone as indicator. It is also possible to weigh the complex as Cd $(C_9H_7N)_2$ $(SCN)_2$.

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References

1. Prodinger, W. Organic Reagents used in Quantitative Inorganic Analysis. *Elsevier Publ. Co. Inc.*, 155, 1940.
2. A. Dolphe-Spakowski and Henry Freiser. Analytical Chemistry, 286, 1949.
3. Vogel, A. I. A Text Book of Quantitative Inorganic Analysis, 264, 1964.